PUZZLE MATH

George Gamov & Marvin Stern

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Prologue: How This Book Came into Being

During the summer of 1956, one of us (GG) was visiting San Diego, California, as a consultant to Convair, where the other of us (MS) holds a permanent position. We had to discuss many (classified) problems together and since the office of one of us (MS), on the sixth floor of the main building, was more comfortable, the other (GG) would usually take the elevator from the second floor, where his office was located. In doing so, one of us (GG) began to notice that when he arrived at the elevator on the second floor and pushed the button, the first car to come would usually be going in the wrong direction; that is, down. In fact, in about five cases out of six, the first car was going down and only in one out of six was it going up.

"Look here," said one of us to the other, "do you continuously make new elevator cars on the roof and send them down to be stored in the basement?"

"What a silly idea!" exclaimed the other, "Of course we do no such thing. I suggest that you check as to how often the correct elevator comes first when you leave this office on the sixth floor to return to the second floor."

Several weeks later, the subject came up for discussion again, whereupon one of us said the first observation was meaningless. In ringing for the elevator at the sixth floor, he had found that five times out of six the first elevator to arrive was going up rather than down. He quickly proposed the reverse of the earlier explanation. It must be that the company was building elevator cars in the basement and sending them to the roof, whence Convair planes flew them away.

"But," exclaimed the other one, "I didn't know that we were in the elevator-car-manufacturing business. ... Of course," he continued, "the real explanation is very simple. But first let me add that if I hadn't known how tall this building was, I could now say from the information you have given me that it must be seven stories high."

"But I never mentioned the height of the building; I was only talking about my problem in getting the correct elevator."

"Yes, but don't you realize this is a classical puzzle which merely illustrates the distinction between frequency and phase?"

After thinking a little, we got the answer to this puzzle (look for that answer in the story called "Passing Trains,", but these conversations revealed the fact that we both were much interested in various kinds of mathe-matical puzzles and between us knew a great many of them.

Hence we decided to put them together in a little book, presenting each puzzle in the form of a short story. The puzzling part of each story is printed in ordinary type, while the part containing the answer is printed in italics. Readers who want to try their skill at solving the puzzles must stop reading when the italics begin. Good luck!

M. STERN

G. GAMOW

Landfall, Woods *Hole,*

Massachusetts

1.

The Great Sultan

TWELVE IN ONE

The great Sultan Ibn-al-Kuz of Quasiababia was sitting in his treasury chamber looking with satisfaction at twelve large leather bags lined up along the wall. The bags con-tained large silver coins representing taxes collected by his emissaries in the twelve provinces he ruled. The name of the emissary of the particular province was clearly written on each bag. Each coin weighed a full pound, and since all the bags were almost full, there was a lot of silver.

Suddenly a man dressed in rags was brought in by the guards and fell on his knees before the sultan.

"Majesty," exclaimed the man, raising his hand, "I have a very important thing to-tell you."

"Speak, then," said Ibn-al-Kuz.

"I am in the employ of one of your emissaries and I want to report his great treason and crime toward you. All the coins in the bag he sent to you are underweight by one ounce. In fact, I was one of the workers who rubbed the coins with rough cloth, taking one ounce of silver from each of them. Since my master wronged me, I have decided to let you know the truth."

"Who is your master?" asked Ibn-al-Kuz, frowning, "I swear to Allah that he will be decapitated tomorrow, and you will get a rich reward!"

"He is . . ." began the man.

At this moment a dagger thrown by an unknown hand whistled through the air, and the man fell dead with the dagger sticking out of his back.

It might seem an easy matter to find which of the twelve bags contained the underweight coins, provided that the sultan owned reasonably precise scales, which could distinguish between regular 16-ounce coins and the molested coins which weighed only 15 ounces. In fact, the sultan did have such scales, and very fancy ones at that. He had had them custom-made by a fine instrument manufacturer in the United States of America, and they were modeled upon the regular weighing machines which one finds practically everywhere in this highly industrialized country. There was a platform to step on and a slot in which to drop a penny and instead of showing the weight on a dial the machine printed a ticket giving the exact weight in pounds and ounces complete with fortune on the back. The trouble was, however, that among all the coins in his possession Sultan Ibn-al-Kuz had only one American copper penny. He could put any assortment of silver coins from all the bags on the platform, but he could get only a single answer on the printed ticket for the total weight of that assortment.

The sultan sat for a long time in deep thought. Suddenly the solution flashed into his mind. If all the coins in all the bags were of the same weight, the scales would always show a number of pounds in whole numbers, no matter what assortment of coins was put on it. However, if one of the coins was a bad one, weighing only 15-*ounces, the scales would record so many pounds and* 15 *ounces, that is, the figure would be one ounce below the nearest integer. If two, three, or more coins put on the scale were bad ones, the figure would be* two, *three, or more ounces below the nearest integer. The sultan rose from his seat and bending over the bags took one coin from the first bag, two* coins *from the second, three coins from the third, and so on, finishing with twelve coins from the twelfth bag. He piled all these coins on the platform of the scales and slipped the penny into the slot. The scales showed so many pounds and* 9 *ounces. Thus there were seven bad coins, and they must have come from the seventh bag. The name written on that bag was Ali-ben-Usur, and Ali's head rolled the very next morning.*

A FAMILY PROBLEM

One day the great Sultan Ibn-al-Kuz encountered a truly-perplexing problem. The supreme vizier was insisting that the sultan pass appropriate laws to control the ratio of men to women in the land's future population. It was argued that since approximately the same number of male children as female children were being born, it was becoming increasingly difficult for men of distinction, but of mode means, to maintain harems of appropriate size.

The sultan, although himself a strong believer in monogamy, could not antagonize either the vizier or the established religion of the land. He puzzled for a while, mumbling incoherently about random sequences. Finally his face broke into a knowing smile, and he said to the vizier, “The solution to the problem is very simple! I will issue a proclamation instructing all the women of the land that they will be permitted to continue to bear children only if these children are girls. As soon as any woman gives birth to her first son, she will thereafter be forbidden to bear any more children. Banishment will be the punishment for disobedience to this law!”

Still smiling, the sultan continued, “Surely this will produce the effect you want. Under this new law, you will see women having families such as four girls and one boy; ten girls and one boy; perhaps a solitary boy, and so on. This should obviously increase the ratio of women to men as you desire.”

The vizier, who had been sitting by very quietly while the sultan explained his proposal, suddenly showed signs of understanding and arose elatedly. Finally he had bent the sultan to his will! He took leave of the sultan and hurried away to spread the news of his personal triumph in molding the future of the land.

The young prince had overheard the discussion and formulation of the new law. With tears in his eyes, he came meekly into the presence of his father.

Appealingly he cried, “O Great Sultan, surely you are not submitting to the whims of this fanatic?”

The sultan chuckled and bade his son come closer to him. “I did not yield to these foolish demands.”

“But, Father.”

“Ho, ho,” laughed the sultan. “Let me explain the true significance of this law that I have decreed. Actually, the law will continue an equal ratio of men to women.”

“But how, Father? I do not understand.”

“Think of it in the following way,” said the sultan. “Assume, for simplicity, that at a common time all the women in the land give birth to their first child. These first-born children will have an even distribution—half will be boys and half will be girls. At *this stage, then, we have kept the ratio of one to one.*

“Now the law requires only that half of the women, namely those who had boys, are no longer eligible for the second round. The remaining half of the women will give birth to a second round of children. In this second round, there will again be an even distribution; that is, there will be the same number of boys as of girls. Therefore, the result of the first and second round together still shows a one-to-one ratio between boy babies and girl babies.

“Once again half of the women in the second round are no longer eligible to continue; that is, the women who have had boys. The women who have had girls continue for the third round. Here again, half the births would be of boys and half of girls.

“So you see that the ratio is maintained. Since in any round of births the ratio of boys to girls is one to one, it follows that when you sum the results of all the rounds, the ratio remains one to one throughout.”(This assumes equal and independent probabilities, which biologically is not quite correct.)

FORTY UNFAITHFUL WIVES

The great Sultan Ibn-al-Kuz was very much worried about the large number of unfaithful wives among the population of his capital city. There were forty women who were openly deceiving their husbands, but, as often happens although all these cases were a matter of common knowledge, the husbands in question were ignorant of their wives' behavior. In order to punish the wretched women, the sultan issued a proclamation which permitted the husbands of unfaithful wives to kill them, provided, however, that they were quite sure of the infidelity. The proclamation did not mention either the number or the names of the wives known to be unfaithful; it merely stated that such cases were known in the city and suggested that the husbands do something about it. However, to the great surprise of the entire legislative body and the city police, no wife killings were reported on the day of the proclamation, or on the days that followed. In fact, an entire month passed without any result, and it seemed the deceived husbands just did not care to save their honor.

"O Great Sultan," said the vizier to Ibn-al-Kuz, "shouldn't we announce the names of the forty unfaithful wives, if the husbands are too lazy to pursue the cases themselves?"

"No," said the sultan. "Let us wait. My people may be lazy, but they are certainly very intelligent and wise. I am sure action will be taken very soon."

And, indeed, on the fortieth day after the proclamation, action suddenly broke out. That single night forty women were killed, and a quick check revealed that they were the forty who were known to have been deceiving their hus-bands.

"I do not understand it," exclaimed the vizier. "Why did these forty wronged husbands wait such a long time to take action, and why did they all finally take it on the same day?"

"Very simple, my dear Watson." The sultan chuckled. "As a matter of fact I expected this good news exactly on that day. My people, as I suggested before, may be too lazy to organize the shadowing of their wives for the purpose of establishing their faithfulness or unfaithfulness, but they have certainly shown themselves intelligent enough to resolve the case by purely logical analysis."

"I do not understand you, Great Sultan," said the vizier.

"Well, assume that there were not forty unfaithful wives, but only one. In this case, everybody with the exception of her husband knew the fact. Her husband, however, believing in the faithfulness of his wife, and knowing no other case of unfaithfulness (about which he would undoubtedly have heard) was under the impression that all wives in the city, including his own, were faithful. If he read the proclamation which stated that there are unfaithful wives in the city, he would realize it could mean only his own wife. Thus he would kill her the very first night. Do you follow me?"

"I do," said the vizier.

"Now let us assume," continued the sultan, "that there were two deceived husbands, let us call them Abdula and Hadjibaba. Abdula knew all the time that Hadjibaba's wife was deceiving him, and Hadjibaba knew the same about Abdula's wife. But each thought his own wife was faithful.

"On *the day that the proclamation was published, Abdula said to himself, 'Aha, tonight Hadjibaba will kill his wife.' On the other hand, Hadjibaba thought the same* *about Abdula. However, the fact that next morning both wives were still alive proved to both Abdula and Hadjibaba that they were wrong in believing in the faithfulness of their* wives. *Thus during the second night two daggers would have found their target, and* two women *would have been dead."*

"I *follow you so far" said the vizier, "but how about the case of three or more unfaithful wives?"*

"Well, from now on we have what is called mathematical induction. I have just proved to you that, if there were only two unfaithful wives in the city, the husbands would have killed them on the second night, by force of purely logical deduction. Now suppose that there were three wives, Abdula's Hadjibaba's, and Faruk's, who were unfaithful. Faruk knows, of course, that Abdula's and Hadjibaba's wives are deceiving them, and so he expects that these two characters will murder their wives on the second night. But they don't. *Why? Of course because his, Faruk's, wife is unfaithful, tool and so in goes the dagger, or the three daggers, as a matter of fact."*

"O *Great Sultan," exclaimed the vizier, "you have certainly opened my eyes on that problem. Of course, if there were four unfaithful* wives, *each of the four wronged husbands would reduce the case to that of three and not kill his wife until the fourth day. And so on, and so on, up to forty wives."*

"I am glad," said the sultan, "that you finally understand the situation. It is nice to have a vizier whose intelligence is so much inferior to that of the average citizen. But what if I tell you that the reported number of unfaithful wives was actually forty-one?"

THE DATE OF THE HANGING

The vizier sentenced Abdul Kasim to be hanged for stealing a loaf of bread.

For a long time the vizier had wanted to execute Abdul and he seized this meager excuse. Unfortunately the law provided for a last-minute appeal to the sultan. This appeal could be made only on the day on which the hanging was to take place.

Knowing that Sultan Ibn-al-Kuz would rescind the sen-tence, the vizier conceived an ingenious scheme to avoid this possibility. In the presence of Abdul, the vizier gave hanging instructions to the warden. The instructions specified the week during which the hanging was to take place, but the warden was instructed to surprise Abdul as to the exact day of the week on which he was to be hanged.

In Abdul's presence the vizier told the warden, "You are to bring Abdul his breakfast every morning. You are to arrange to hang Abdul on a day of the week such that he could not have pre-computed the day, in order that he be completely surprised. If, when you bring Abdul his breakfast on the morning of the day that you have chosen for the hanging, he confronts you with the statement that he knows he is to be hanged that day and offers a rational explanation of how he figured it out, then the law requires you to allow him to bring his appeal immediately to the sultan's attention. If, however, when you bring Abdul his breakfast on the morning of the day chosen for his hanging, he has not been able to figure rationally that this would be the day, he loses his privilege of appeal and you can hang him that afternoon."

The vizier departed, and Abdul and the warden retired to try to compute the most probable day for the execution. The problem was complicated for the warden by the fact that, in order to make the necessary preparations, he had to determine the date in advance.

This nefarious scheme to have Abdul effectively denied the privilege of appeal by virtue of the surprise aspect in the sentencing came to the ears of the young prince, who in-formed his father.

The sultan summoned the warden before him.

"I hear," said the sultan, "that the vizier has sentenced Abdul to be hanged and that the sentence was so designed as to obviate the possibility of Abdul's putting in a formal appeal. Is that not so?"

"Yes, Great Sultan," replied the warden, "but please believe that I had no part in this. In fact, I am very fond of Abdul and would be happy to do anything possible to help him, but, as you can see, my hands are tied."

"Well," said the sultan, "I understand that you are a very learned man."

"Oh no, Great Sultan, I do read the teachings of the wise men and I am very fond of the logic that they teach, but I am really only a student of these teachings and have yet a great deal to learn."

"That is most interesting," replied the Sultan, "Do you understand the principle of finite induction?"

"Oh, yes, Great Sultan."

"Well, let us leave that for a moment and reconsider poor Abdul. Have you chosen a day for the hanging?"

"No, not yet, I am still considering it."

"I understand," the sultan went on, "that you can hang him only on a day of a specified week which starts on a Sunday and ends on Saturday. I wonder now, Could you wait until the last day of the week, Saturday, to hang Abdul?"

The *warden thought a moment and slowly replied, "No, I guess I cannot. You see, Great Sultan, Abdul is an intelli-gent man. Since he knows the stipulation of the hanging sentence, if he were alive Saturday morning when I brought his breakfast to him, he - would say to me, 'I know, warden, that you must be planning to hang me today since today is the last of the days upon which I can be hanged.'*

"And that would be true," continued the warden. "If he were still alive on Saturday morning, my last opportunity to hang him, he would most definitely know he was to be hanged that day, and he would confront me with this very fact. He would thereby have recourse to an appeal."

"I see," mused the sultan. "I *take it then that both you and Abdul must realize that, according to the stipulation of the sentence, it would therefore be impossible for the hanging to take place on Saturday."*

"Yes," agreed the warden, "it must be that I can only hang Abdul on one of the days, Sunday through Friday. Saturday is not admissible for the hanging."

"Then," continued the sultan, "you and Abdul can cross Saturday off your calendars. You could hang him only Sunday through Friday, and Friday then is the last of the admissible days upon which Abdul could be hanged."

"Yes," replied the warden,

"I wonder then," asked the sultan, "could you wait until Friday?"

After thinking again, the warden replied, "I know I cannot. Since Abdul and I would both know that Friday is the last day admissible for the hanging, if Abdul were alive on Friday morning when I came in, he would confront me with the statement that I was going to hang him this day."

"Yes, I see your reasoning," replied the sultan. "Do you .mean then that you can only hang him from Sunday through Thursday and that Abdul must *realize this?"*

"Yes, definitely" replied the warden with assurance.

"If that is the case then," continued the sultan, "and Thursday becomes the last of the admissible days for the hanging, can we not continue the same argument and thereby exclude Thursday?"

"Why, of course" exclaimed the warden, "and that would leave Wednesday as the last admissible day, which would thereupon exclude itself in turn—and so on back through the entire week. The hanging, therefore, is impossible!

"This is truly an application of the principle of finite in-duction. What we have effectively proved is that the last of the admissible days for the hanging automatically becomes inadmissible."

"Yes," said the sultan, laughing, "And because of this, one can successfully argue the impossibility of the hanging forward through any finite number of days. But it's lucky that you and Abdul are of equal intelligence; for if either of you jailed to see it, it would not work out in practice."

ABDUL'S SECOND ORDEAL

After the narrow escape described in the previous story, Abdul again got into trouble. This time he was accused of having dealings with the black market in women slaves, which had recently been outlawed by Sultan Ibn-al-Kuz. On this occasion Abdul had to be tried by a jury, an innovation introduced by the progressive potentate who was striving to transform his country into a modem state. The jury of six men and six women split over the verdict; all six women considered him guilty and demanded capital punishment, but the six men held out for "not guilty."

Thereupon the judge decided that Abdul should be given a fifty-fifty chance of life or death, the outcome to be decided by drawing a ball from a bowl. For this unusual method of rendering judgment the court provided two large bowls with twenty-five white balls and twenty-five black ones in each. The prisoner was blindfolded, and then had to stretch out his hand, choose one of the bowls, and select one ball. A black ball meant death; a white one life. Of course the bowls had to be shuffled, and the balls in both thoroughly shaken, after the blindfold was put on.

"O great Judge," exclaimed Abdul, falling on his knees in front of the bench, "grant me one last request! Permit me to redistribute the balls between the two bowls before I am blindfolded and have to choose the bowl and the ball."

"Do you think this could improve his chances of escaping the punishment?" asked the judge, turning to the vizier, who was sitting next to him.

"I don't think so," said the vizier, who considered himself a great expert in mathematical problems. "There are fifty black balls and fifty white ones, and, since he cannot see them, the chances remain the same no matter how the balls are distributed between the two bowls—or among any number of bowls."

"Well then," said the judge, "since it will not make any difference, why don't we grant his request, if only to show our great sultan that his newly appointed court of justice has liberal tendencies in accordance with his wishes?

"Go ahead; redistribute the balls," he said to Abdul, who was still kneeling before him.

Rising *and approaching the table Abdul stretched his hand to the bowls. His procedure was quite a simple one. First he poured all the balls from one of the bowls into the other, and then, selecting one white ball from the lot, he put it back into the first bowl. This ingenious rearrangement increased his chance for life to nearly* 75 *per cent. After being blindfolded, he had a 50-per-cent chance to choose the bowl containing the white ball, and, if he got the* wrong *bowl, he still had* 49 out *of* 99 *chances (that is, almost* 50 *per cent) to get a white instead of a black ball.*

History does not record whether this increase of favorable chances enabled Abdul to save his life.

HORSE RACE IN REVERSE

One sunny day, as all days are in that part of the earth, an Englishman was sitting on a rock in the middle of a desert in the domain of Sultan Ibn-al-Kuz. He was bored, since there was absolutely nothing to do, even though he had enough money in his pocket to pay for any kind of entertainment. Thus when to his great pleasure he saw two Bedouins riding by, he signaled them to approach.

"Friends," said he, holding out a shining golden guinea, "I would like you to race to that palm over there, and I will give this golden coin to the one of you whose horse comes in last."

"Whose horse comes in last!" exclaimed the Bedouins, who both knew English.

"Exactly so. I realize that it is an unusual condition, but that is what I said. Now start."

Desirous of money, the two Bedouins started toward the distant palm, but since each of them was trying to hold his horse back, they made very little progress. When they were almost ready to give up the race, a dervish appeared unexpectedly in front of them, and, jumping off their horses, they prostrated themselves before him in the hot desert sand.

"What is the trouble, my sons?" asked the dervish in a low voice, and they explained to him the condition of the race.

"Maybe we should just split the money, or decide between ourselves that he who wins the race by having his horse come last, will give his winnings to the other," one suggested.

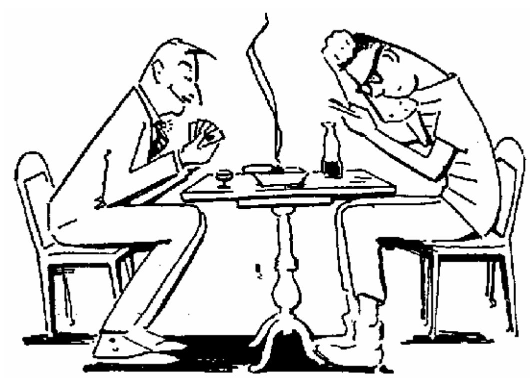
"Oh, no!" said the dervish. "One should be honest in all deals, even with Englishmen. But here's what you can do." And he whispered his advice to them.

"Allah bless you!" exclaimed the Bedouins, jumping into the saddle and putting their spurs to the horses' sides.

They galloped to the palm faster than the wind. The race was decided in a few minutes and the Englishman had to pay a guinea to the winner. What did the dervish say?

It is *very simple indeed: he advised them to exchange horses.*

2.
Gamblin' Sam



CARDS IN A HAT

Perhaps the simplest way to describe Gamblin' Sam is to say that he was a Damon-Runyon-type character.

His ability to earn an honest buck was assured by his ability to figure the odds in any "game of chance" that had ever been invented by man. This, of course, was completely a function of his memory rather than a result of any creative analytical ability on his part.

Another essential characteristic of Gamblin' Sam was that he had a good heart He had promised Sam Junior's mother that Junior would be brought up respectable. Sam stuck to this promise. He reasoned that Junior should have inherited some of his own calculating ability; and what could be more respectable today than being a mathematical scientist?

So Junior was sent to college. True, he might spend the rest of his life commuting to work on the bus, rather than driving a Cadillac as Sam did. And although Junior would possess many books, he might never have a little black one with the name and address of every chorus girl in town. But Junior was going to be respectable.

Gamblin' Sam was extremely proud of his professional ability. In talking with Junior during his senior year at school, Sam was elated to find that many of the calculations employed in modern physics used probability techniques. This brought forth a lengthy dissertation by Sam, who wanted to show off his great experience in figuring odds.

Junior tried to explain that there was really a great deal more to probability theory than Sam realized. Much had been developed with rather sophisticated mathematical techniques. Even in the simplest cases of figuring the odds, in which Sam was most expert, Junior questioned whether Sam really understood the basic principles of probability.

"For example," said Junior, "consider the following game. I put three cards into a hat. One card is red on both sides, one card is white on both sides, and the last card is red on one side and white on the other.

"Assume that I draw one card. This card I draw comes up with a red side showing. We don't know what the other side is."

"Am I to guess what the other side is?" asked Sam.

"Yes," said Junior. "You are to tell me what the odds are of this card having red on the other side. You see," he con-tinued, "if the card that I have drawn has a red side showing, then it must be one out of two possible cards either the red-red card, or the red-white card. Isn't that right?"

"Yes, that's right"

"Well, then what would the odds be that it is the red-red card?" asked Junior.

"Well," said Sam, disgusted at the simplicity of the problem, "since there are only two possible cards that you could be holding, the odds are merely one out of two that the card is the red-red one."

"I knew that was what you'd say," said Junior. "But it's not the right: answer'"

"For this, your mother wanted me to send you to college!" exclaimed Sam. ''Don't think you can teach me about odds. This doesn't involve any mathematics; it's merely common sense."

"You *see'' Junior continued patiently, "the question merely involves a correct definition of the term 'probability.'* Your *answer that it is one out of two cards and, there-fore, the resulting odds would be merely one out of two disregards the conditions of the problem. I stated that* I *had drawn a card with a red side showing. In order to evaluate the probability that this card is the red-red card, I must first ask myself how many ways I can draw a card with the red side showing"*

"Yes, that's right," said Sam, "but why should this change the answer?"

"Well," continued Junior, "among *the cards placed in the hat there are a total of three red sides. That is, there are two red sides from the red-red card and one red side from the red-white card."*

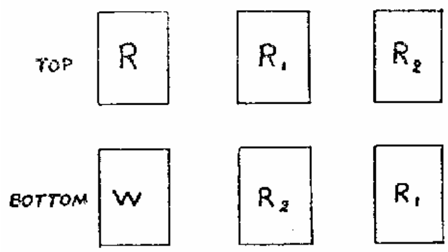
"Of course," said Sam.

"Then you agree that there are three possible ways of drawing a card with red showing on top."

"Yes, yes, I agree"

"Now let's look at these three possible ways of drawing a card with red showing on top. One way, the card would have white on the other side; that is, if I had drawn the red-white card. The other two ways must, *therefore, have red on the other side; that is either way of having drawn the red-red card. So you see that of these three possible ways of drawing a card with red showing, two ways would have red on the other side and only one way would have white on the other side. So the odds that the card I hold is a red-red card must be two out of three."*

"Well," *mused Sam, "you are a pretty fast talker, but I am not sure that 1 am convinced."*



"I *can prove it another way" said Junior. "If you were witling to play this game with me on the assumption that I had drawn the card with red showing, then we could just as well have played it on the assumption that I had drawn a card with* white *showing."*

"Yes, there's no difference" said Sam.

"But in this new game," continued Junior, "if I had drawn the card with red showing, I would have asked what were the odds of this being the red-red card, and if I had drawn a card with white showing, I would have asked what were the odds of this being the white-white card. Here you can see the true problem. The game is the same, whether played on white or red. Playing the two games simultaneously (red or white) must therefore yield the same resulting probability as playing either game alone, but when both are played at once the solution stands out. What I am really asking is: out of the three cards, what are the odds of pulling a card which is the some on both sides as compared to pulling a card which is different on the two sides? And here, of course, the answer is two to one, as there are two cards out of the three which are the same on both sides."

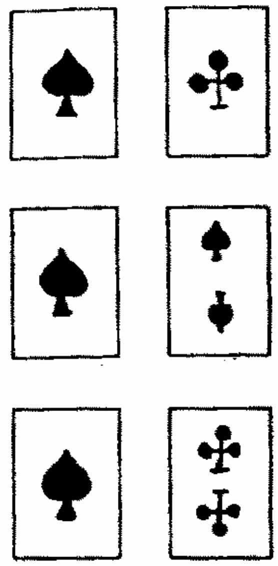
ACES

"Well," Sam said defensively, "you've constructed a freak problem. I am sure that the need to use such a formal defi-nition of probability in figuring simple odds never arises in practice. Besides, nobody plays games with cards like that anyway!"

“I am not so sure it doesn’t” said Junior, “I can give a similar example using ordinary cards.”

“All right, let’s try that one.”

“Here we go then. Assume that you have dealt a bridge hand. In this hand, I find an ace of spades, and of course there are twelve other cards in the hand. These twelve others are completely random.”

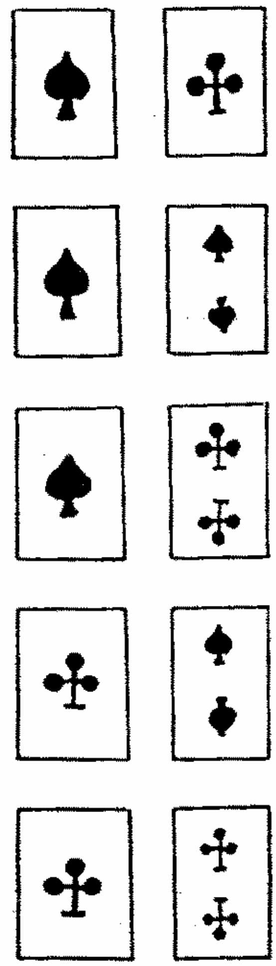


And, *of course, the probability of drawing the hand with the two aces out of the three possible hands is one out of three,”*

“That’s right” said Junior. “Now what are the possible hands that could be dealt with an *ace guaranteed?”*

“Well, that’s simple, too,” said Sam.

“Let’s see now” continued Sam, “that gives us five possible hands, and out of these five, only one of them has two aces. This leaves us with a probability of only one out of five! But why should this be?”



Junior laughed, and explained, "We define the probability for success simply as the number of successes out of the total number of trials. Now, in both the problems we've discussed, it is the total number of possible trials that is deceiving."

"In the simplified problem, the restriction to the particular ace, the ace of spades, has resulted only in reducing the total number of possible hands. It hasn't altered in the least the number of these hands that are successes. Of course, in the extended problem of the bridge hand, the numerator, or rather the number of successful hands will be restricted by the restriction to a particular ace, but the total number of hands possible with an ace of spades will be much more restricted. The probability in this case will be higher than it is when merely a random ace has been guaranteed."

RANDOM PROBABILITY

"Well," sighed Sam, "you are beginning to convince me. Maybe I had better go back to matching pennies, or something."

"Oh, I wouldn't go so far as to say that. But you have reminded me of an interesting story. In school last year, we were all forced to take a foolish course which contributed absolutely nothing to our education. It was an old require-ment that had apparently never been eliminated. The instructor himself was very apologetic about wasting our time. As a consolation, though, he announced at the beginning of the semester that he would give us all A's or B's, so we needn't worry about grades, but only about wasting time.

"But this instructor believed very strongly in being fair. This created a little problem for him when he had to assign grades at the end of the term. He was going to give A's and B's. But he wanted to distribute them at random among the students so that each one stood the same chance of

"He thought he would do this by going down the list of students' names and tossing a coin at each name to give either an A or a B. But before he did it, he got a horrifying thought—what if the coin were a little weighted! There would then be a pre-set bias, and he wouldn't be assigning grades fairly.

"His problem then was how, with a possibly weighted coin, he could assign grades of A or B at random so that each student had an exactly equal chance of receiving either grade."

Sam chuckled, and said, "I always knew that was the way grades were given, but I never thought of the need to eliminate the effect of a possibly weighted coin. I think I see how to do it, though. What if he tossed the coin twice? Isn't it true that, regardless of the bias, the probability of first getting a head and then a tail is exactly the same as the probability of first getting a tail and then a head?"

Junior laughed too. "That's right! And of course if both tosses gave the same, you would disregard them and try two more. It would count only when he tossed first a head and then a tail, or first a tail and then a head; and then he would give the A or the B according to which had come up first.

"The reason this gives the correct result, though, is very interesting," continued junior, ''and I would like to explain that.

"Assume that the probability of getting a head on any one toss is p. *Then the probability of getting a tail is* (1 — p). *It follows that the probability of first getting a head and then a tail, out of two tosses, is the product of these numbers, or p (1 – p).*

"In *the same way then the probability of first* getting *a tail and then a head would be* (1 - p) p.

"But since *ordinary multiplication is commutative that is, in the product of two* numbers, if doesn't *matter* which *number is considered to be the multiplier and which is considered to be multiplied – these two* products *are equal:*

$$p\,(1-p) = (1-p)p$$

And *that is why your answer is* right.

MATCHING PENNIES

Sam smiled and said, "I knew that when it came to money, I'd be able to show you that I know my business."

"I am sure you do," replied Junior. "Actually, I was only trying to bring out some rather elementary fine points. As a matter of fact, in your field you do more than just worry about probabilities. You do some game theory and, in effect, you compute strategies."

"I do no such thing!" exclaimed Sam. "I just have a lot of experience in my work, that's all."

"Well, you may do it intuitively. But I am sure that you do engage in what we call game theoretic techniques. Let me show you what I mean by applying a formal analysis to a very simple example.

"Assume that we are going to match pennies. If my penny and your penny have the same side showing, you win. If they are different, I win. But we can make this game more interesting.

"If you win by head-head, I will pay you nine cents. It you win by tail-tail, I will pay you only one cent.

"On the other hand, for either way that I might win, that is, either tail-head or head-tail, you must pay me a

"You can stack your coins beforehand in any manner that you want. In fact, as the game proceeds, you may have the privilege of changing your stacking.

"You can see that this will add a lot of interest to the game of matching pennies. This game allows for good strategy in playing. Obviously since you win most on head-head, you might prefer to stack your corns with a predominance of heads. Bat since I know this, I may decide to stack mine with a predominance of tails so that I will win.

"So each of us is faced with the problem of figuring out the best scheme of stacking his own coins knowing that the other fellow is working out a scheme for himself.

"That does sound interesting," said Sam. Since the average of what I could win each time is the average between nine cents and one cent, and that is the same as what you could win each time, I guess the game is fair enough to play this game with you, and I am sure that I can stack my corns so as to outsmart you and teach you to have a little respect for your old man."

Junior shook his head. "I'm sorry but I won't take your money. You see, this game is rigged. I can use a strategy in stacking my coins that assures that in the long run the best you can hope for is to lose as little as possible. But you will lose, and I will win. I can compute mathematically what proportion of the time I should play heads regardless of whether you play heads or tails. And the computation will tell me how much I can win over a long period of time.

"I'll give you the mathematics, though you may have to take it on faith. It's a very interesting proposition. The way it works is this:

"I want to compute what proportion of the time I should play heads. Let us call this proportion x. *Then let us consider how my payoff will vary with* x. *We will call my pay-*off P.

"First consider what happens when you play heads. Every time I play heads when you play heads, I lose nine cents. Since I play heads x part of the time, this means that my payoff function has in it a term —9x. Similarly every time I play tails when you play

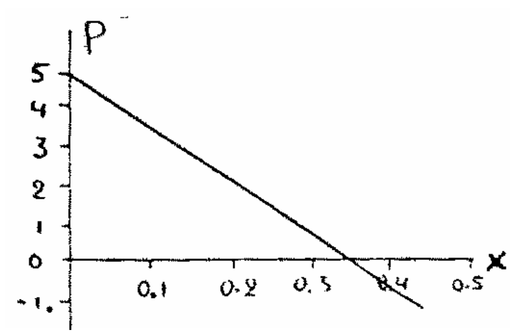
heads, I win five cents. Since I play tails (1 — x) part of the time, my payoff function has in it a term + 5(1 — x).

"So *if I write my complete payoff function for the occa-sions that you play heads, it comes out to be*

$$P_H = 9x + 5(1 - x)$$

"This is *simplified into:* $P_H = -14x + 5$

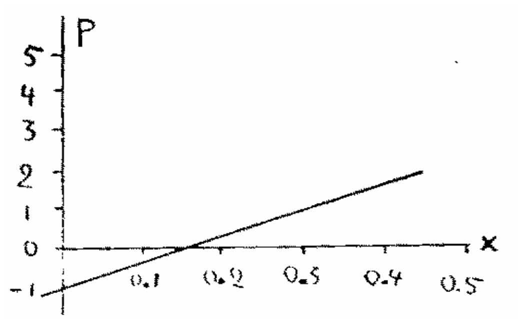
"The plot of this is shown here.

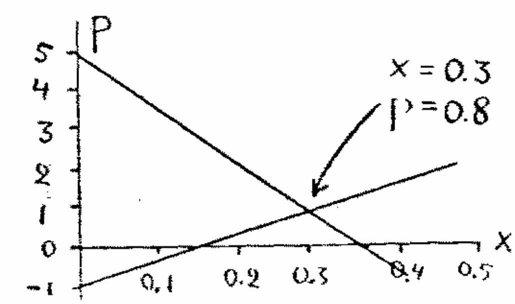


"Now *let us consider what happens when you play tails. Constructing my payoff function in the same manner as before, I* get

$$P_T = +5x - 1(1 - x)$$

$$P_T = 6x - 1$$





"Superimposing these two plots we find an intersection at x = 0.3, P = 0.8.

"This means that if I stack my coins with three-tenths of them heads, distributed at random through the stack, over a long enough period of time I will win an average of 0.8 cent for each coin we match."

BIRTHDAYS

That certainly is a shrewd one, though I don't pretend [to understand why it works," said Sam. 'I'm going to try it out at the club tonight. By the way, I wonder if you could recommend some other hustles I could pull on the fellows there."

"Yes, I have a very nice one," said Junior. "First tell me how many men you are expecting tonight."

"Oh, about thirty," replied Sam.

"Fine, that will be a good number for this birthday riddle I know. If you recorded the birthday of each of the men there, what odds would you give me against finding a duplication? Remember when 1 say birthday, I mean the month and day—not the year of birth."

"Well," said Sam, "out of thirty men chosen at random, I guess the odds should be something like twelve to one, but I will only give you five to one against a duplication of birthdays."

"Good," said Junior, "I will gladly take that, and I suggest that you try it on some of the fellows there. If anyone offers you better than even money against the duplication, you take the bet."

"This I certainly do not understand," exclaimed Sam.

"It's just another sample of what we call 'the multiplicative nature of independent probabilities," continued *Junior. "In this case, you continue to ask for birthdays, as long as you haven't already achieved a duplication, until you have gone through all thirty men as the limit.* So, *since this process continues only on failure, the probabilities that must be multiplied are the probabilities of failure on each questioning of a new man. The probability for success, of course, is 1* minus *the resulting probability for failure.*

"In other words, the second man whose birthday you record has 364/365 chances for not duplicating the first man's birthday. *When it comes to the third man, he could duplicate either of the first two men's birthdays, so his chance for not duplicating either of them is 363/365*

"This means that after you have asked three men their birthdays, the probability of not getting a duplication within this group of three is 364/365.363/365 and, of course, the probability for getting a duplication is 1 - 364/365.363/365. So you *see that in going through the whole group of thirty men, the probability of getting a duplication is: 1 - 364/365.363/365.....336/365.*

"There are various ways of evaluating this number. At any rate, the answer to this one is that the probability for getting a duplication is about 0.7, which means, of course, that you have greater than two-to-one *odds for getting a duplication."*

"That is truly amazing," said Sam. "How many men would one have to ask to make it about an even money bet that a duplication in birthdays exists[7]"

"That works out at about twenty-four. It is interesting that the odds for duplication go up very rapidly is you *add birthdays beyond twenty-four."*

TENNIS TOURNAMENT

"I guess that's enough problems on probability for a while," said Sam. "With the material you have given me, I should be able to clean up for several weeks to come. I understand that you are going to play a lot of tennis this summer and forget about all this math,"

"Yes, you are right," answered Junior, "but strangely enough, I find myself with a problem in tennis that I cannot solve easily with math."

"Mathematics in tennis!" exclaimed Sam. "What do you mean?"

"Well," said Junior, "it's not really mathematics in tennis; it's just that I am running a tournament of junior tennis players and I can't seem to figure out how many cans of tennis balls should be provided for the tournament. You see, in running the tournament, we take all the contestants and pair them off for the first round; after the first round of matches, we take the winners and pair them off for the second round. We .continue this until we finally end up with a single winner.

"The problem is that I have to supply a new can of tennis balls for each individual match between two players. Now, if at any round in the tournament there is an odd number of men, one man draws a 'bye' and does not compete in that round. He then gets into the next round if possible.

"What is confusing my calculations is the possibility of having these odd men drawing byes at one round or another. For a given number of initial entrants in the tournament, and allowing for the possibility of the men drawing a bye, I cannot seem to figure out the total number of matches that will be played."

Sam laughed. "Well, this is one that I can help you on. Just ignore completely the fact that there may be an odd number of men at any round. Instead of trying to analyze the number of matches that will be played round by round and allowing for the byes, it is much simpler to look at the over-all picture. After all, each match eliminates one man. So if you start with n men, since of course you must end with one man who is the winner, n - 1 men must be eliminated. That means that n - 1 matches must be played order to eliminate this number of men. Therefore you must be prepared to supply n - 1 cans of tennis balls."

ONE-SIDED GAME

Once Gamblin' Sam and his mathematically minded had a bet on some minor question, and Junior suggest that, instead of the usual states of a few dollars, the loser should play the winner a game to determine how much he had to pay.

"It's a very simple game," said Junior, "Just tossing a coin. If, for example, you lose our bet, we toss the coin if you win the toss, it's all over and you owe me nothing. If, on the other hand, you lose the toss, you give me dollars and we toss a second time. If you win in this again the thing is over, and all I win from you will be two dollars. If, however, you lose the second time, you give me an additional four dollars, and so on—you pay me double each successive time you lose. The game continues as long as you lose the toss, and stops as soon as you win for the first time. Is it a deal?"

“It’s a deal,” said Sam, the gambler’s instinct rising in him. “I’m sure that, even if I lose the bet with you, I have a fifty-fifty chance of paying you nothing, and even if I lose the toss at first, it won’t be long before I win.”

The next day Sam was found to have lost his bet, and so he had to play the coin-tossing game to determine how much he should pay.

“Listen, Dad,” said Junior, “why don’t we just estimate mathematically what you owe me instead of actually tossing the coin? I’m willing to settle for that if you are. After all, you wanted me to study mathematics, so why not let me use it?”

“Okay,” agreed Gamblin’ Sam reluctantly, although he would have much preferred to toss the coin. “Let me see you make a mathematical estimate of my debt to you, and I’ll pay up if it’s correct.”

“Oh, it’s very simple, and you will understand it even without much knowledge of mathematics. On the first toss, I have equal chances to get nothing or two dollars. Thus, it would be fair on my part to ask you for one dollar instead of tossing.”

“Fair enough” said Sam.

“All right, how about the second game, in which I’m to get four dollars if I win? There’s one chance in two that this game will be played at all, since we play it only if you lose the first one. And, if we play it, there’s one chance in two that I’ll win and get four dollars from you Thus the chances are only one out of four that I get this four dollars if we toss. Therefore, I’m entitled to a quarter of four dollars, or one dollar for not playing the game.

“Hm,” said Gamblin’ Sam worriedly, “I’ll have to pay you that dollar for not playing the second game. But what about the third game? Will it also cost me a dollar

“But of course,” said Junior. “I can win the eight dollars in the thud game only if it is played, which will happen only if I win the first two; this is a one-in-four probability Besides, I have only a fifty-fifty chance to win, we play it, so that the total probability of getting eight dollars only one in eight. And the same argument holds for every succeeding game, so that I can ask you for a dollar each for an infinite number of tosses. You haven’t got that much in the bank. But I’ll be kind to you; give me just ten thousand bucks, which I need to buy a new sports car.

“Why, you young grafter!” exploded Sam. So that s the way you are using your education! Well, you’ll have to wait until tomorrow; maybe I can find a flaw in your argument.

All afternoon Sam went over and over the arguments presented by his son, but he could not find anything wrong. It really looked as if he would have to pay a dollar for each succeeding game and thus give Junior all the money he had. Driven practically to despair, he remembered that the club he managed was frequented by a short gray-haired man who was said to be a retired mathematics professor.

Isn’t my boy smart, thought Sam with mixed feelings: that old fellow is always losing money despite of all his mathematical knowledge?

Finding the professor’s name and address in the club’s debt-record book, Sam knocked on the door of his apartment the same evening, “I have a proposition for you,” he told the professor. “I’ll cancel your outstanding debts to the club if you will help me with a mathematical problem.” Then Sam explained the predicament he was in.

“Oh, that’s very kind of you,” said the professor, rubbing his hands. “This trick your boy used on you is known as the St. Petersburg paradox and was invented by the famous German mathematician, Leonard Euler, who was at that time serving at the Russian Imperial Court. May I ask how much money you have altogether in your bank account?”

“But I don’t want to pay all that money to the boy!”

“You won’t have to; but I have to know that figure in order to estimate how much you have to pay. The point is this: although it is perfectly true that for each succeeding game you have to pay one dollar, you should count only as many games as you could actually play before you lost all your money and became bankrupt.

“If you play n *games, losing-all the time, the amount of money you would have paid to your son is given by the sequence:* $2 + 4 + 8 + 16 + …. + 2^n. *It is a simple geometrical progression, and its sum is equal to 2 multiplied by itself the number of times you play plus* 1, *from which you subtract 2 at the end. Mathematically, this is*

$$2\,[\,(2^n - 1)/\,(2 - 1)] = 2^{n+1} - 2$$

This expression increases very rapidly with the number of games played. Thus, if you lost the first ten games, the amount you would have to pay your son would be $2^{n+1} - 2$, *which in this case is* $2^{11} — 2$, or $2046. *I suppose you have more than that in your account?”*

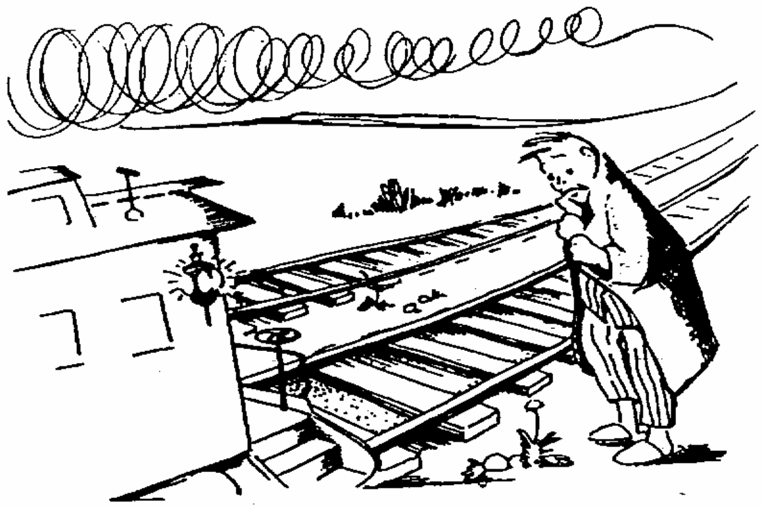
“Oh yes, I have about half a million dollars” said Sam frankly.

“Well, in order to lose half a million dollars in that game, you only need to lose eighteen tosses in succession—you would not have enough money to pay if you lost the nine-teenth time. Thus, your boy could count on only eighteen successive winnings, and you owe him eighteen bucks. Give it to him tomorrow.”

“Thanks a million!” exclaimed Gamblin’ Sam. “Too bad he’s too old for me to give him eighteen snaps of my belt along with it. Good night, sir. I’ll be happy to extend you more credit at the club.”

3.

The Railroad Man



PASSING TRAINS

In a small Midwestern town there lived a retired railroad engineer named William Johnson. The main line on which he had worked for so many years passed through the town. Mr. Johnson suffered from insomnia and would often wake up at any odd hour of the night and be unable to fall asleep again. He found it helpful, in such cases, to take a walk along the deserted streets of the town, and his way always led him to the railroad crossing. He would stand there thoughtfully watching the track until a train thundered by through the dead of the night. The sight always cheered the old railroad man, and he would walk back home with a good chance of falling asleep.

After a while he made a curious observation; it seemed to him that most of the trains he saw at the crossing were traveling eastward, and only a few were going west. Knowing very well that this line was carrying equal numbers of eastbound and westbound trains, and that they alternated regularly, he decided at first that he must have been mistaken in this reckoning. To make sure, he got s little note-book, and began putting down “E” or “W,” depending on which way the first train to pass was traveling. At the end of a week, there were five “E’s” and only two “W’s” and the observations of the next week gave essentially the same proportion. Could it be that he always woke up at the same hour of night, mostly before the passage of eastbound trains?

Being puzzled by this situation, he decided to undertake a rigorous statistical study of the problem, extending it also to the daytime. He asked a friend to make a long list of arbitrary times such as 9:35 a.m., 12:00 noon, 3:07 p.m., and so on, and he went to the railroad crossing punctually at these times to see which train would come first. However, the result was the same as before. Out of one hundred trains he met, about seventy-five were going east and only twenty-five west. In despair, he called the depot in the nearest big city to find whether some of the westbound trains had been rerouted through another line, but this was not the case. He was, in fact, assured that the trains were running exactly on schedule, and that equal numbers of trains daily were going each way. This mystery brought him to such despair that he became completely unable to sleep and was a very sick man.

The local doctor whom Mr. Johnson consulted concerning his health was also an amateur mathematician and a collector of puzzles.

“This is a new one to me,” said he, when Mr. Johnson described the cause of his troubles. “But wait a minute, there must be a rational answer.” And, after a few minutes of reflection, the doctor had the answer ready.

“You see,” said he, “the whole thing depends on the fact that the trains are running on a schedule, even though you were arriving at the crossing at any odd time. Let us sup-pose that eastbound trains pass our town every hour on the hour, whereas the westbound ones pass by at a quarter past each hour. There are, of course, an equal number of trains in each direction. But let us see which train will be first to pass when you arrive at the crossing. If you arrive between an even hour and a quarter past the hour, say between 1:00 and 1:15 p.m., the first train to pass will be that going west that is, the 1:15 p.m. train. However, if you arrive after 1:15 p.m., and thus miss that train, the next one will be at 2:00 p.m., going east.

“If you go *to the crossing at random times, the chance that you arrive during the first quarter hour is three times smaller than the chance of arriving during the remaining three-quarters of an hour. Hence, the probability that the first train which passes by will be going eastward is three times greater than the probability that it will be going west. And that is exactly what you have observed.”*

“But I don’t understand. If the probability of an east-bound train is three times that of a westbound train, doesn’t it follow mathematically that there must be more east-bound trains?” objected Mr. Johnson. “I don’t know much about mathematics, but it seems to be a natural conclusion.”

“No,” said the doctor with a smile, “don’t you see? The first train to pass is most likely to be eastbound, because the chance of your arriving during the period between a west-bound and an east-bound train is three times as great. But you will have a much longer average wait in that case”

“How so?” exclaimed the puzzled engineer. “What do you mean by a longer wait?”

“Well, you see,” continued the doctor patiently, “if you come to the crossing during the first quarter hour, so that the first passing train is a westbound one, you will never have to wait for it more than fifteen minutes. In fact, the average waiting time will be only seven and a half minutes. On the other hand, if you have just missed the westbound train, you will have to wait for almost forty-five minutes before the eastbound train comes. Thus, although the probability that the first train will be eastbound is three times larger than otherwise., the time you have to wait for it is also three times larger, which makes things even.

“It may not be exactly a quarter of an hour against three-quarters, but I’m sure you will find, if you check the schedules, that this is the general pattern. Given an equal number of trains alternating in each direction, this is the only way your observation could be true over a long period. There must be a shorter interval from each eastbound train to the next westbound one than there is from each westbound to the next eastbound”

“I must think about it,” said Mr.” *Johnson, scratching his head. “So you say it is because of the schedule?”*

"Well, one can also put it another way, without referring to the schedule," said the doctor (This latter explanation applies directly to the elevator problem of the prologue.)

"Let us take, for example, a single train, the Super chief, passing through here between Chicago and Los Angeles. We are about five hundred miles from Chicago and fifteen hundred miles from Los Angeles. Suppose you come to an intersection at any odd time. Where, most probably, is that train?

"Since the track from here to Los Angeles is three times longer than that to Chicago, the chances are three to one that the train is to the west rather than to the east of you. And, if it is west of you, it will be going eastward the first time it passes. If there are many trains traveling between Chicago and California, as is actually the case, the situation will, of course, remain the same, and the first train passing our city after any given time is still most likely to be an eastbound one."

"Thank you," said Mr. Johnson, rising and reaching for his hat, "I think you have cured me even without any pills."

OVERLAPPING TRAINS

A few days after his visit to the doctor, Mr. Johnson received a telephone call from him.

"If you can come down to my office this afternoon," the doctor said, "I would like to ask you another question about railroads."

"Of course," said Mr. Johnson, who, being retired, had plenty of spare time.

"Here is a problem," said the doctor when Mr. Johnson walked in, "which one of my patients gave me when I told him about your experience with the westbound and eastbound trains. When he drives to work, he has to cross the single railroad track used mostly by freight trains, which are notoriously long and move very slowly going through the city. Quite often he has to sit in his car in front of twinkling red lights and wait while what seems an endless procession of freight cars slowly drags by. He told me he wished it were a double track, in which case the eastbound and—westbound freights would sometimes overlap, thus shortening the total waiting time at the crossing. Do you think that if-that were the case, he would really have to spend less time waiting?"

"Of course," said the old railroad engineer. "If the total number of trains using the double track is the same as those using a single track, the occasional overlapping of two trains going in the opposite direction should cut down on the average waiting time. Why! If two trains are passing the intersection simultaneously, the time of waiting for two rains reduces to that for only one."

"This is true, but only in the case when they overlap exactly. However, in another extreme case, the situation is mite different. Consider the case when the locomotive of one train enters the street-crossing just as the caboose of another train is leaving it. What then?"

Well, I reckon that is no different from the case when the trains do not overlap at all."

Oh, no. You *are quite* wrong *here, and I can prove it to you by simple arithmetic. Suppose there is on the average one train per hour each way, and that each passing train* *blocks the intersection for six minutes. Let us calculate the waiting time in that case. There is one chance in ten that the motorist will arrive at the crossing while the train is passing, or rather while red signals are blinking.* Now, *since it is equally probable that he will arrive at the intersection when the train is just entering or just leaving, the average waiting time for a passing train is three minutes. Thus, the mean waiting time will be* 0.3.

"Now, *suppose that the trains always overlap but just slightly—the locomotive vis-à-vis the caboose. This is, of course, equivalent to having half as many trains which are, however, twice as long."*

"Makes no difference" commented the railroad engineer.

"Oh yes, it does! The chances that you run into a red light arriving at the intersection are, of course, the same. But, if you are *stopped by red lights, your average waiting time is twice as long. Thus the trains overlapping just on engines would make the situation for motorists twice as bad."*

"I see," said the railroad engineer thoughtfully. "Indeed, if there were just a few minutes interval between the two trains, the motorists which were waiting for the first one to pass could go through, while in the case of overlapping, the gap is closed."

"I am glad you see the point" said the doctor. "So we come to the conclusion that in the case of exact overlapping, the mean waiting time is cut in two, while in the case of just barely overlapping, the waiting time is doubled."

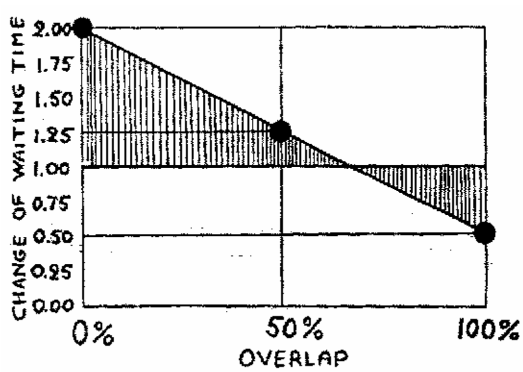
"Let us see," suggested Mr. Johnson, "what happens if trains overlap half and half."

"We can do it too. This case will be equivalent to having *half as many trains, each train being fifty per cent, that is, by a factor* 1.5, *longer. In this case, the chance of taming to the intersection while a train is passing must be multiplied by factor 1.5/2, and the mean waiting time if this happens increases by factor* 1.5. *The total chance of mean waiting time will be given by the product 1.5 /2 x 1.5 = 1.125.*

Thus in case of half-overlapping, the waiting time will be increased by twelve and a half per cent"

"Even *for a half-overlap, it is still unfavorable," said* Mr. *Johnson with surprise.*

"'Yes, and that is significant. Suppose we make a graph showing the result of complete overlap, half overlap, and edge overlap," said the doctor, taking a pencil.



"You *see from this diagram that the total area which corresponds to the increase of mean waiting time is considerably larger than that corresponding to a decrease. Thus,* *we must conclude that, on the average, the overlapping of passing trains will make the motorists at intersections wait longer than in the case of the same number of trains traveling both ways on a single track."*

THE BUMBLEBEE

"Do you know any more train problems?" asked the rail-road engineer, after a short silence.

"Yes, I know one more, but it is very simple. Two trains start simultaneously toward each other from two stations A and B, one hundred miles apart, both traveling at the speed of fifty miles per hour. A bumblebee starts at the same time from station A toward station B flying along the railroad track at the speed of seventy miles per hour. When it encounters the train advancing from B, it gets scared, turns back, and flies toward A. And so it flies to and fro between two advancing trains until the trains finally meet. When it sees the two trains onrushing from both sides the bumblebee is so scared that it drops dead.

"Question: What is the total distance flown by the bumblebee?"

"'Well, let us see" said Mr. Johnson. "It does *not seem to be very difficult. If both trains are advancing at fifty* miles *per hour from two stations located one hundred miles apart, they must meet in the middle one hour after starting.* How *fast did you say the bumblebee was flying?"*

"Seventy miles per hour.

"Well then it must have flown just seventy miles. Isn't it true?"

"Good for you!" exclaimed the doctor. "But the fact that you got the solution so easily shows that you are not a mathematician. A really good mathematician would think about it in terms of an infinite series representing the sum of the time intervals necessary to cover each swing in the 'oscillatory motion of the bumblebee. It becomes a rather horrible mathematical exercise since the algebraic expressions for these time intervals are rather complicated. I heard that when this problem was given once to the late Doctor John von Neumann, who was one of the greatest mathematicians of this century, he thought just for a fraction of a minute and gave the correct answer of seventy miles.

"'Oh!' *said the man who gave him this problem. 'So you got the simple-minded solution- I thought you would sum the infinite series in your head.'*

" 'But I did sum the infinite series,' said John von Neumann, who was known to be able to carry out complicated calculations in his head at a speed which was second only to that of electronic computing machines, toward the development of which he made major contributions."

THE CARRIER PIGEONS

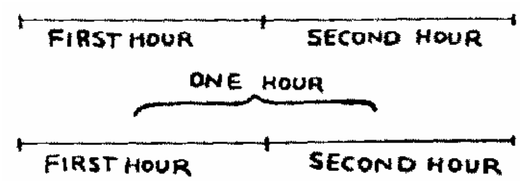
Mr. Johnson was describing to his mathematically inclined friend the doctor some of the difficult requests he had re-ceived when he was working as an engineer. The Signal Corps was once very eager to conduct a test on carrier pigeons and had asked him if, on one of his regularly sched-uled train runs, he would dispatch two carrier pigeons exactly

50 miles apart and exactly one hour apart in time. Now it was true that there was one run wherein there was a straight-line track for 100 miles. The schedule was such that a train would cover this 100 miles m exactly 2 hours, thus achieving an average velocity of 50 miles per hour for these 2 hours. However, there were numerous stops and starts over "this 100-mile distance, and the exact time consumed by each stop depended somewhat on the passenger traffic getting on and off the train. The railroad engineer was able to make up for lost time by accelerating to higher speeds occasionally, and he always succeeded in making the entire 100-mile run in the required 2 hours.

"But," he said to his friend, "just because I cover a hundred miles in two hours there is no reason to presume that within these two hours there must always exist a one-hour interval over which my average velocity is fifty miles per hour."

"Unfortunately, you are wrong." The doctor laughed. "Regardless of the variations in speed during these two hours that cover one hundred miles, it is easy to prove that there must always exist at least one conserve one-hour *interval of time, in which you cover exactly fifty miles. Perhaps the simplest way to see this is as follows: Consider the two hours divided up into two one-hour adjacent intervals.*

"Now we further assume that you do not cover exactly fifty miles during *either the first hour or the second hour. Otherwise the problem is immediately solved. We can also assume that during the first hour the average velocity is less than fifty miles per hour, and during the last hour the average velocity* is *greater than fifty miles per hour.* You *will see that the argument I am going to present is really independent of which hour has the greater average velocity.*



"Next let us consider a one-hour interval by itself and let us move this one-hour interval of time continuously along the scale, starting with a complete overlap with the first hour and moving it along until it finally completely overlaps the second hour.

"For this one-hour interval which we are going to slide along the scale, we will consider the average velocity achieved. Of course since this sliding one-hour interval starts by completely overlapping the first hour, the average velocity of the sliding interval is initially less than fifty miles an hour. We then slide the one-hour interval along until it finally overlaps the whole second hour. Now *the average velocity over the sliding one-hour interval has become greater than fifty miles per hour.*

"By sliding the one-hour interval, we find that the aver-age velocity over this one-hour interval varies continuously from less than fifty miles an hour to greater than fifty miles an hour. It follows, therefore, that somewhere in the sliding process the average velocity over *this one-hour interval must pass through a point at which it is exactly fifty miles per hour. Thus the proposition is proved."*

The railroad engineer sighed and said, "I guess you are right, although it wouldn't have helped the Signal Corps much, because I could never know in advance when the train would reach the beginning of that stretch, so I wouldn't have known when to release the pigeons. But while I think of it, I have a very practical problem in which you may be interested."

"Go ahead," said the doctor, "although I'm not usually very good at practical things."

DAYLIGHT SAVING

"As you know," Mr. Johnson said, "I was on the same run for many years. Every evening I arrived at my home town at exactly the same time, and I turned the train over to any other railroad man who took it on from there."

"Yes, I know this," said the doctor.

"Since railroad schedules are so perfect, my wife knew precisely at what time I would arrive, and she drove the car down to the station to pick me up. Every night she arrived just as I pulled in, and took me home."

"A most cooperative wife," said the doctor.

"Well," the engineer went on, "I recall one year when there was a disagreement among several states as to whether they should go on daylight-saving time. As it worked out there was no change of time in my home state, but since my run originated in an adjacent state which went on day-light saving, the schedule was upset and I arrived at my home-town station on the first night after the change exactly one hour earlier than usual according to our town clocks. I suddenly realized that my wife wouldn't be there until-exactly one hour later. I also remembered that the engineer who took over the train from me had broken his y repeatedly changing it back and forth one hour, and I had lent him mine.

"There I was with exactly an hour to wait, so I decided I had better start walking down the road toward home. I walked and I walked and finally I met my wife who was driving along the road toward the station. She picked me up and we drove home. When we arrived, I looked at the clock on the wall and saw that with all my walking, I had arrived home only twenty minutes earlier than usual. Although I did not have my watch with me, 1 know 1 must have been walking for a very long time. As a matter of fact, I tried to figure out how long I had been walking and I was stymied. Perhaps you could tell me how to figure this out."

"Yes, certainly I can" said the doctor. "The simplest way to figure this one out is to forget about yourself and merely think about your wife's trip that night. If we do that, then we know that your wife started at the regular time but arrived back home twenty minutes earlier than usual. In order for her to have saved twenty minutes on the round trip to and from the station, she must have saved ten minutes each way. In other words, she must have picked you up at a time which was ten minutes before she would have arrived at the station if she had continued. But she would have arrived one hour after you had started to walk. This means you were walking for the difference between that ten minutes and one hour; in other words, you were walking for fifty minutes."

4.
Travel Is So Broadening

THREE SOOT-SMEARED FACES

As is well known to travelers in Europe, the passengers on British trains habitually keep the car windows open to get fresh air, and along with the fresh air they often get a good deal of smoke from the locomotives wh.ch, according to British custom, use coal as fuel. Thus it is easy to imagine that some smoke blew through an open window into a train compartment occupied by three well-bred Britons who were sitting stiffly on their seats minding their own business. As the result of that accident, the faces of all three passengers were smeared in spots by black soot, which presented an amusing contrast to their impeccable clothes and snobbish bearing. One of them, a lady called Miss Atkinson, raised her eyes from the book she was leading, and in spite of her perfect upbringing could not help chuckling at the sight that met her eyes. The two men were chuckling also.

But-and this may be characteristic of British nature-each of the three passengers assumed that his or her face was clean, and that the two others were chuckling at the sight of each other's faces. (Good manners prohibited them from looking directly at the object of interest, so that it was impossible to see who was chuckling at what.)

This situation lasted for a few minutes. Then Miss Atkinson, who was better educated than the other two, being a school teacher, suddenly realized that not only were the faces of the other two passengers smeared with soot but also that her own face must be. She took out a handkerchief and rubbed her face thoroughly from ear to ear and from fore-head to chin. An inspection of the handkerchief proved that her conclusion was correct, but how did she come to that conclusion?

It was not too difficult, being based on a reasonable assumption by Miss Atkinson that her two co-travelers, while perhaps not as intelligent as she was, were, however, not complete morons.

"Assume," thought Miss Atkinson, "that my face is dean, as I hope it is, and that these two gentlemen are chuckling because each sees the other's face soot-smeared. If this is really so, the man sitting on the right must see only one soot-smeared face—that of the other man. What, in that case, would be his train of thought? He would naturally ask himself why the other fellow is chuckling, and, since he sees that my face is clean, the only reason for that chuckling is that his own face is smeared too. The conclusion is elementary, and since neither of them came to it my original assumption that my face is clean apparently does not hold. Thus I'd better clean my face in order not to appear ridiculous."

Miss Atkinson s logical conclusion can be generalized for the case when there are more than three passengers in the compartment, with all their faces smeared by soot. In *fact, the fourth passenger, presumably of still higher logical ability, would argue this way:*

"If my face is clean, the three other passengers with smeared faces must be laughing at one another. But at least one of them must be intelligent enough to realize that if his face were clean, the remaining two passengers must be laughing at each other, and, not being complete morons, must realize that their faces are smeared"

And so on and so forth.

A KISS IN THE DARK

During the Nazi occupation of France four passengers were riding in a compartment of a train rolling from Pans to Marseilles. It was a rather odd company—a young and very good-looking girl, a dignified-looking old lady, an officer of the German occupation forces, and a middle-aged Frenchman of indefinite profession. None of them knew any of the others, and no conversation was started while the train was speeding south. When the train rolled into a tunnel, the electric lights in the car failed to function, and for several minutes the four passengers were submerged in com-plete darkness.

Suddenly there was a sound of a kiss, followed almost immediately by the sound of the hard impact of somebody's fist against somebody's face. When the train came out of the tunnel, the original position of the passengers was unchanged but the German officer had a large bruise under his eye.

"Serves him right," thought the old lady. "Here is a true French girl who will defend herself against the aggressiveness of these Boches! I wish we had more such brave and chaste girls."

"Strange taste this German officer must have," thought the French girl. "Instead of kissing me, he chose to this old bag. I cannot figure it out!"

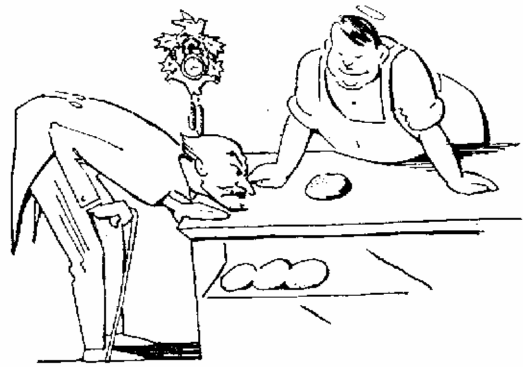
The German officer, holding his hand against his injured eye, could not figure out the situation either. The best explanation he could make of what had happened was that the Frenchman had tried to kiss the young girl and that she had struck out in the darkness and accidentally hit him in the face.

The problem is to find out what the Frenchman thought, and what actually happened.

Although some readers may protest that this problem cannot be classified as a mathematical puzzle, its solution is however, unique. The Frenchman, who was a member of an underground resistance group, was very proud of himself. He was sure that clever as Germans are this one would never figure out that the Frenchman had just kissed his own fist, and then smashed it into the German's face.

BREAD RATIONING

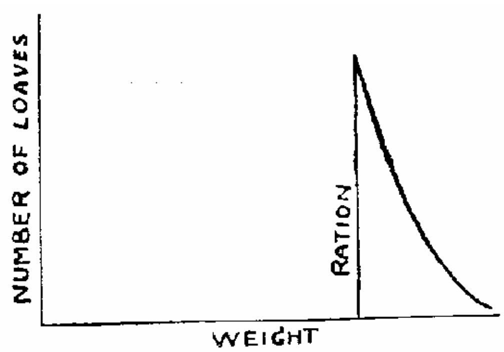
After Germany had lost the war, the economic situation of the country rapidly deteriorated. All food was rationed, the most important item, bread, being limited to two hun-dred grams per person per day. Every baker was instructed to get special forms in which to bake loaves of exactly that size, one loaf daily for each customer in his district.



Dr Karl Z., an old physics professor, stopped at the bakery each morning on his way to the university to get his daily “lion. One day he said to the baker in his district, “You are a bad man and are cheating your customers. The forms you are using are five per cent smaller than they should be for baking 200-gram loaves, and the flour you save you are selling on the black market.”

“But Herr Professor,” exclaimed the baker, “nobody can bake loaves of bread of exactly the same size. Some will be a few per cent smaller than prescribed, some a few per cent larger.”

“Exactly so,” replied Professor Z. “I have been weighing the loaves you have given me for the past few months on exact scales which I happen to have in my laboratory. They show a natural variance. But here is a graph of the number of your loaves of different weight as compared to the correct weight:



“You see that, whereas some loaves weigh as little as 185 grams and others as much as 205, the *average* weight obtained in mean measurements is 195, instead of 200. You must get new forms of the correct size, or else I will report you to the authorities.”

“I will certainly do it tomorrow, Herr Professor,” said the scared baker, “and you may rest assured that this mistake will not be repeated.”

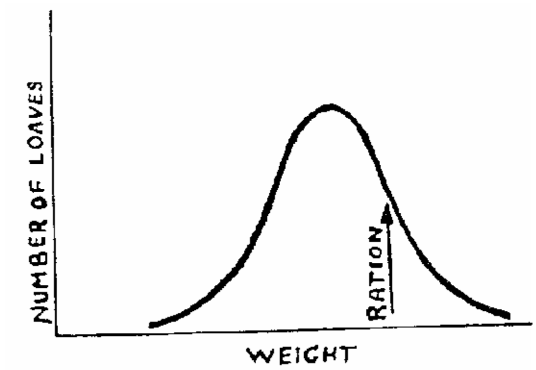
A few months later Professor Z. addressed the baker again. “I have reported you to the authorities today,” he said. “You didn’t change your forms and are continuing to cheat your customers.”

“But Herr Professor!” exclaimed the baker. “You cannot accuse me of cheating now. Were any of the loaves I gave you during these last few months underweight?”

“No, they were all 200 grams or over. However, this did not happen because you got larger baking forms, but be-cause you were specially selecting the larger loaves for me, giving the underweight ones to other customers.”

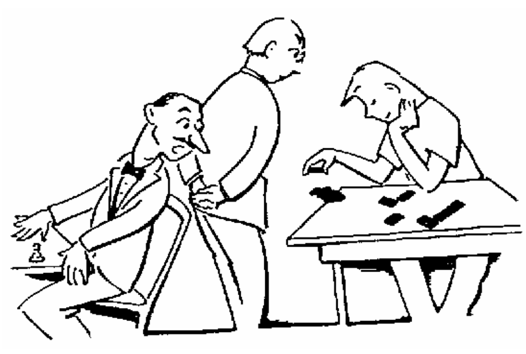
“You cannot prove it!” said the baker arrogantly.

“Oh yes I can” declared Professor Z, “Look at the statistical distribution I got by weighing your bread for the last few months:



“Instead of the standard error distribution *demonstrated by the great German mathematics, Karl Fredrich Gauss, we have an oversized curve, sharply cut on the left side, and slowly descending on the* right *Statistical deviations from the mean cannot possibly lead to such a distribution, and it is clear that it was produced artificially by selecting loaves weighing 200 gm or more. It is just the tail of Gauss’s distribution and that is the same distribution I obtained by weighing loaves before our previous conversation. I am sure the rationing authorities will take my word for it.”*

5.
Young Nicholas



DOMINO GAME

Young Nicholas was very eager to become a member of the men’s Chess and Checker Club in his town. Unfortunately he was considered much too young. It seemed obvious to the distinguished members of the club that such a young man could not have the proper imagination, over-all ability to visualize patterns of play, and other necessary qualities.

This belief they held firmly in spite of the fact that young Nicholas was a highly proficient player both of chess and of checkers. One particular evening young Nicholas, as usual, was found at the club watching the masters perform. In reply to his humble request to play a game, he was mockingly told to find himself a set of dominoes and learn that children's game.

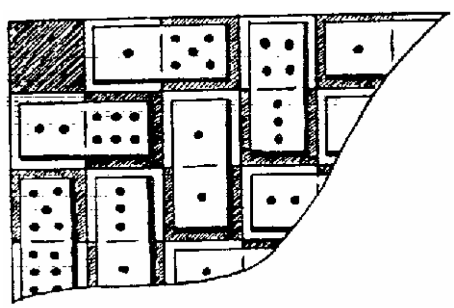
"But," appealed young Nicholas, "I cannot see how this would prepare me for chess and checkers."

"Well, go and practice dominoes for a while—maybe it will help."

Several days later, young Nicholas remarked to one of the elders that in playing with dominoes, as he had intended to do, he had found an interesting problem relating dominoes to the checkerboard.

With many smiles, the elders of the group condescended to listen.

"As we all know," Nicholas began in a very authoritative manner, "a checkerboard is composed of sixty-four squares, eight squares along each side. If we try to cover that checker board with dominoes, since each domino consists of rectangle two-squares long and one square wide, we would need thirty-two dominoes to cover the entire checkerboard. Suppose, however, that we have only thirty-one dominoes. No matter how you place the dominoes on the checker board two squares of the board are bound to remain empty. Now suppose I start putting the dominoes on the board in such a way that the square in the upper left-hand corner is left empty, and continue to place dominoes in a compact pattern until all thirty-one are used. Evidently one other square of the checkerboard must be left empty. The problem is to place the dominoes in such a way that the empty square is the one in the lower right-hand corner.



"I guess this is one of those problems of visualization of form in which you gentlemen are such masters."

The elders looked at one another and agreed that it was an interesting exercise. They then proceeded to go to work on the problem. They took thirty-one dominoes and labori-ously tried pattern after pattern to see if they could find one which would leave the lower right-hand corner empty.

Several nights later, they informed young Nick that it was impossible to leave that square empty, and so the problem was solved.

"But," asked Nicholas, "How do you know it isn't possible?"

"Well," exclaimed one of the elders, "we have tried every possible pattern and none of them works, so it must be impossible."

"I guess that is the right way to put it, but you have not explained why it is impossible."

"Why, because we can't do it," they declared.

"I still wonder if the answer to the question could not be put in a more satisfactory form."

"What do you mean by a more satisfactory form?" they asked.

"Well," *continued young Nicholas, "I would look at it in the following way. Since this checkerboard has equal numbers of black and white squares, and since each domino covers one black square and one white one, the two leftover squares must be of different colors. But the diagonally opposite squares are of the* same *color,* so *that there is no possible way you can leave them both empty. It is an interesting case of a problem in which the introduction of a condition which at first glance seems irrelevant simplifies the solution. Actually, all that is necessary for the formulation of the problem is a grid of eight by eight squares without any coloring at all. But, in order to solve the problem, we have to divide the squares into two groups, and these might as well be called black squares and white ones. And that makes the solution straightforward and easy!"*

BUILDING-BLOCKS

One of the elders was so highly impressed by Nicholas's reasoning as to suggest that perhaps he should be allowed to play checkers. Another was very emphatic in his refusal, insisting that Nicholas should stick to children's games. He even suggested building-blocks in his scorn. In response, the more sympathetic elder remarked, "That reminds me of a problem using dominoes as building-blocks, in which you gentlemen might be interested."

"I don't think we have time to worry about making building-blocks out of dominoes," replied one of them in disgust.

"Still, why don't you listen to this? Perhaps you'll enjoy it," urged the first man.

"Consider a case wherein you have access to all the dominoes that you want. This problem is to pile them one on top of the other to build a column with as much offset as you can. You are free to offset each domino at one end as little or as much as you please, but the final column must be stable enough so as not to topple."

Several of the men immediately began to guess how large an offset one could build in this manner. These guesses ranged from one-half domino length to possibly full domino length.

"No, those are not the correct answers," Nicholas's cham-pion said with a smile.

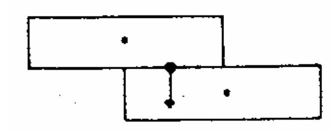
"Well, how much of an offset could you build?" he was asked impatiently.

"Oddly enough," he answered, "you could build an offset as large as you please."

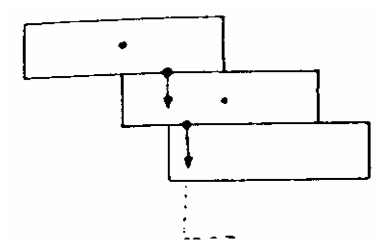
"I don't believe it," they exclaimed in unison. "You will have to prove this."

"What do you think, Nicholas?" asked his friend.

"If is *very simple" said young Nicholas. "You can analyze the stability of the final column by starting at the top and working down. The maximum offset between the top* *domino and the one directly beneath it* is, *of course, half the length of a domino, so that the center of gravity of the top domino falls right on the edge of the domino beneath it.*



"So *now we have only a one-half domino-length offset built in. Now let us consider where the center of gravity of the top two dominoes would be. If we try the two dominoes as shown, on top of a third, we find that the combined center of gravity is one-fourth of a domino's length in from the covered end of the middle one. So, of course, we could place the top two dominoes on the one beneath them with this one-fourth additional offset.*



"Now if *we computed the center of gravity of the top three dominoes, we would find it to be one-sixth of a domino*'s *length in from the edge of the bottom or third* domino. *If we continue this process, we find the total offset that can be built is as follows:*

$$\frac{1}{2} + \frac{1}{4} + 1/6 + 1/8 + \ldots\ldots,$$

right out to infinity"

'Is that *mathematically correct?" one of the elders asked of the man who had posed the problem and who happened to be a mathematician,*

"Certainly," he replied. "The formula can be rewritten as:

$$\frac{1}{2}\,[\,1 + \frac{1}{2} + 1/3 + 1/4 + \quad + \ldots\ldots]$$

"Now this bracketed term is known as harmonic series, and diverges. By this I mean it could be made larger than any pre-assigned number. The easiest way to see this is that each group could be made larger than one-half.

"We group the series as follows:

$$1 + \frac{1}{2} + (1/3 + 1/4) + 1/5 + (1/6 + 1/7 + 1/8) + \ldots..$$

Each group is now seen to be greater than 1/2; *that is,*

$$1/3 + 1/4 \text{ is greater than } \frac{1}{4} + \frac{1}{4} \text{ (i.e., } 1/2);$$

$$1/5 + 1/6 + 1/7 + 1/8 > 1/8 + 1/8 + 1/8 + 1/8 = 1/2;$$

and so on.

"So you see if you specify how much of an offset you want to build, it is possible to compute how many dominoes would be needed to build this offset according to the scheme young Nicholas has proposed. My computation is from the top down, but of course the building would be from the bottom up."

DIAGONAL STRINGS

The members of the Chess and Checker Club were fasci-nated by the problems of the checkerboard and the pile-up of dominoes into an offset column, and looked eagerly at one another for more in the same vein. Young Nicholas ventured to say, "I have another one in which you gentle-men might be interested."

"Go ahead, the floor is yours," he was told. By this time they were converted.

Suppose there were one nail in each of the four walls of this room and in addition one nail on the ceiling and one in the floor. Suppose further that we had to be strings between these nails. Each nail is connected to each or the other nails by a separate string. I have two colors of strings which I use—red and blue. Each connection between any two nails is either with a red string or with a blue siring.

"All these strings make up many triangles; that is, any three nails can be considered apexes of a triangle formed by the strings connecting these three nails. My problem then is to see if I could distribute the colors so that no triangle has all three sides the same color."

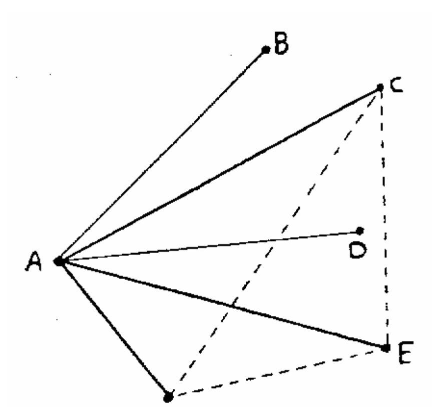
"This is rather complicated," mused the mathematician, "It must involve calculations of permutations and combina-tions, and the like. I didn't think you knew that much algebra, Nick."

"I don't," replied young Nick, "but I can still do the problem."

"Well, all right, tell us how," said the elder.

"It is really very simple," said young Nicholas. "You only have to know enough to start reasoning.

"The *answer is that there will be at least one triangle all sides of which are the same color, I will show that it is impossible to avoid this.*



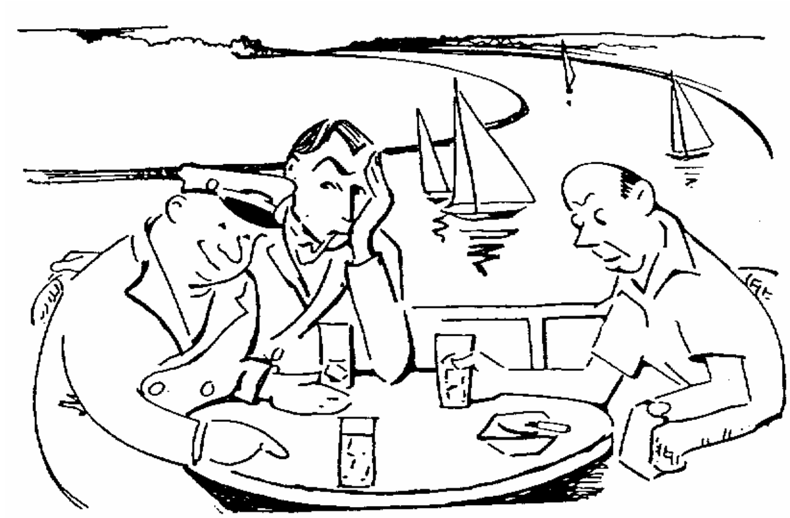
"Consider any one nail. Out from it there must stretch five strings, one to each of the other five nails. No matter how you distribute the colors in these five strings, at least three of them must be the same, since you have only two colors. For the sake of argument let us assume that three of the strings are red.

"Consider now the triangle formed between three nails at which the three red strings have terminated.

"If we are to try to avoid a triangle with all three sides the same color, it follows that these three nails cannot all be joined to each other with one color. Putting this *more simply, the triangle formed by the three terminal nails should not be all blue.* At *least one of the* strings *between the three terminal nails must be red. But if* so, we *have completed a red triangle from the original nail."*

6.

The Yacht Club



WINDLESS SAILS

One hot and windless summer afternoon, a group of mem-bers of the River Yacht Club were sitting on the club's veranda sipping gin and tonic.

"Can't do much sailing without wind," said one of them philosophically.

"Sometimes you can," said another. "In fact, I once sailed for quite a distance without any wind at all."

"Without any wind at all?"

"Yes."

"And using your sail?"

"Yes, using my sail."

"Did you blow into your sail, or what?" asked the first skeptically.

"No, I used my sail in quite the conventional way. But I'd better explain to you what the circumstances were. I was in a little sailboat in the middle of the river when the wind suddenly died out. I did not have any oars or other means of propulsion, and my sailboat was just drifting downstream. Then I noticed about a hundred yards ahead of me a small rowboat with the oars flapping on both sides, but with nobody in it. If I could get this rowboat, I could tow my sailboat to where I wanted to go. But how could I cover the hundred yards? Since there was absolutely no wind, both boats were floating downstream at the same speed, and the distance between them remained constant."

“So what did you do?”

“Try to guess.”

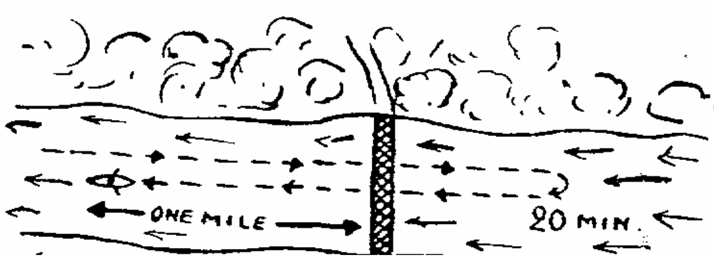
“Well, I don’t think it is possible to move downstream faster than the river if there is no wind,”

“And yet it is possible. When I said there was absolutely no wind, I meant, of course, that the air was motionless in respect to the ground. Since, however, my sailboat was moving downstream with the river, there was a slight breeze in respect to my boat, blowing in the direction opposite to that of the river’s flow. The situation was the same as if I were on a lake, and a slight wind was blowing from the direction of the rowboat which was standing motionless. So all that I had to do was tack against this wind until I reached the rowboat.”

BOAT AND BOTTLE

“The answer to the rowboat problem sounds like Einstein’s theory of relativity,” remarked one of the yachtsmen. “Relative motion, yes, but not the theory of relativity by a long shot,” said another who had read some popular books on the subject. “But it reminds me of another story in which it is important which coordinate system you choose for the description of the phenomena. There was a man in a boat rowing upstream on a river, and on the bow of his boat stood a half-full bottle of good bourbon. When he was passing under a bridge, the boat rolled slightly, and the bottle fell into the water. Without noticing, the man continued to row upstream, while the bottle naturally drifted downstream. Twenty minutes later, the man noticed that the bottle was gone, turned around (you may neglect the time necessary for this maneuver), and started down-stream to recover the bottle. Being a phlegmatic fellow, he continued to row with the same effort, so that, whereas his velocity in respect to the shores had previously been the difference between the velocity of his boat and the velocity of the river, it was now the sum of the same quantities. Well, he finally sighted the bottle, and picked it up one mile below the bridge. Can you tell me, on the basis of these data, what the velocity of the river was?”

Several members of the club who were amateur mathe-maticians tried their hands at the problem, and one of them even wrote an algebraic equation which contained the two unknown quantities: the velocity of the boat in respect to water and the velocity of the river. But neither the direct approach nor the use of algebra seemed to achieve a solution, and they finally came to the conclusion that there was simply not enough data.



"And *yet the solution exists, and is in fact very simple" said the man who had proposed the problem. "All you have to do is consider the problem from the point of view of a coordinate system moving along with the river. From this point of view, the river becomes a lake while the shores and the bridge are moving in respect to it. If you drop something into the water while rowing in a lake, and turn around to recover it twenty minutes later, it takes you just another twenty minutes to come back to the spot. Thus, the bottle was in the water for forty minutes and during that time the bridge moved in respect to the water by one mile. Therefore, the velocity of the bridge in respect to the water, or, what is the same thing, the velocity of the water in respect to the bridge and the shores, was one mile in forty minutes, or one and one-half miles per hour. Simple, isn't it?"*

"But you can't estimate the velocity of the boat that way," said the member who had tried to use algebra. "There are two unknowns in this problem."

"Yes, but the velocity of the boat in respect to the water is not pertinent to the problem, and I didn't ask you to find it. Your difficulty in solving the problem algebraically resulted from the fact that you tried to find two unknowns having only one equation. Actually, the second unknown cancels out, but the equation looks so complicated that you

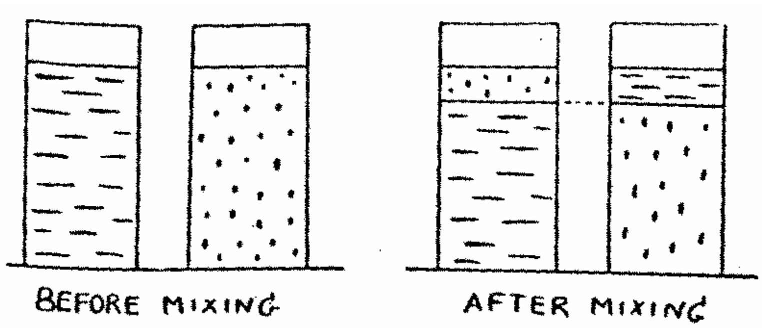
GIN AND TONIC

"Here is another problem for you/' said one of the yachts-men, sipping his gin and tonic. "Suppose you have two glasses filled to the same level, one with straight gin and another with pure tonic water. Now you pour a jiggerful of gin from the first glass and put it into the second. Mix well, and then bring a jiggerful of the mixture from the second glass back into the first. Question: As the result of these two operations, will there be more gin in the tonic glass or more tonic in the gin glass?"

"Of course there will be more gin in the second glass," said another member, "since you brought a jigger of pure gin into it and took back a jigger containing some tonic along with some gin."

"I see you are quite sure about it, but you are also quite wrong. Try to think of it as follows: You brought one jigger of fluid from the first glass into the second, and then one jigger back into the first. Since the glasses were originally equally full, they will be again equally full after these two operations. Thus, the gin which is missing from the first glass is replaced by tonic water taken from the second glass and is present in the second glass replacing tonic water taken from it. The amounts are obviously equal.''

"I still cannot see it."



"Okay. Let's try it in numbers. Say the glasses originally held three ounces each, and you are using a one ouncer jigger. You take away one-third or one ounce of gin and add it to the tonic glass, which now holds four ounces of liquid. This is three-quarters tonic and one-quarter gin. If you mix them evenly and take away one ounce, the jigger also con-tains three-quarters tonic and one-quarter gin. When you put it into the gin glass, the balance is restored, but you have 2 1/4 *ounces of gin and* 3/4 *ounce of tonic. Left in the tonic glass is* 3/4 *ounce of gin replacing the tonic taken away. See it?*

"If it worked out the way you thought, and you got more gin in the second glass than tonic in the first, you would have increased the total, amount of girt and decreased the total amount of tonic. A fine way to turn water into wine!"

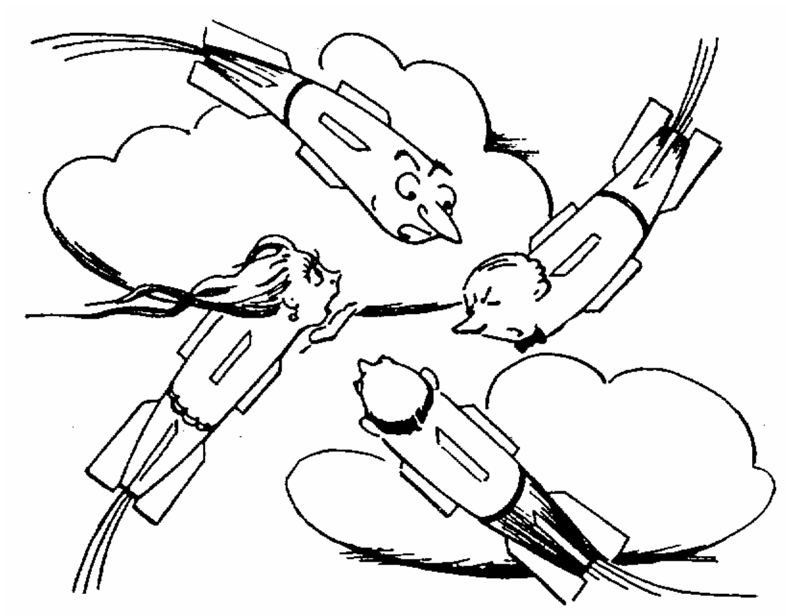
THE BARGE IN THE LOCK

"Getting back to sailing problems," interposed another yachtsman, "here is a nice one for you. I have given it at different times to several physicists, and none of them has given a correct answer. A barge loaded with scrap iron was floating in a canal lock. For some unknown reason, the people on the barge started to throw scrap iron overboard and continued until the barge was entirely empty. The question is: What happened to the water level in the lock?" "Of course it didn't change," said one of the yachtsmen.

"No, the water level must have gone up," insisted another.

"Those are exactly the answers I got from the physicists," said the first speaker. "But, as a matter of fact, both answers are wrong. The water level in the lock went down. You see, according to Archimedes law, any floating object displaces a volume of water equal to its weight. Thus, since iron is much heavier than water, the volume of water which it displaces when it is afloat in the boat is much larger than the volume of that iron. However, when it is in the water at the bottom of the lock, iron displaces only the amount of water equal to its volume. Consequently the water level in the lock must go down when iron is thrown overboard." "It isn't quite clear to me," protested one of the listeners. "Well, think about it this way. Astronomers tell us that some stars, such as the companion of Sirius, are built of matter a million times denser than water. A *cubic inch of that material would weigh several tons. If such a heavy cubic inch is placed in the barge, the barge will sit deep in the water, and the water level will rise. If, however, that cubic inch rests on the bottom, it will displace only one cubic inch of water, that is, practically nothing, and the water level will go down. In the case of scrap iron, the result will be the same, but less drastic."*

7
Aeronautica



AGAINST THE WIND

A group of USAF officers were sitting in an airfield canteen sipping coffee and studying the latest comic strips.

“Look here, Jack,” said one of them, “didn’t you plan to fly to N-base this afternoon and be back for dinner?”

“I have changed my plans,” said Jack. “The base is directly east from here and there is a strong easterly wind which would slow me down. I prefer to wait until tomorrow when calm weather is predicted.”

“But if you plan to return the same afternoon, the wind would make no difference in your flying time,” said the first. The wind is not likely to change before sundown, and on the way back it will be on your tail. That means you will make up the time you lost on the way over.” “Is that so?” drawled Jack. “Sure it is. How else could it be?” ‘You haven’t had much experience in flying,” said Jack. “And you haven’t ever studied the theory of relativity.” “What has it to do with the theory’ of relativity?’” “Well, it just happens to be the basis for an experiment Michelson used in trying to detect the supposed ‘ether wind’ caused by the motion of the earth through space. . . .

“But first let’s settle the problem of my flight to N-base. In case your mathematics are rusty, I’ll try to explain it first in everyday terms. Just remember that the reduced speed due to going against the wind makes you take longer to get there, and the tail wind coming back speeds you up so you take less time to get back. This means that the resistance or slowdown has a longer time to affect you, and the speed-up has d shorter time. So the loss is greater than the gain. Simple, isn’t it?

*“But of course if you got all A’s in math, you won’t be convinced until you see it in a formula. Call the velocity of my plane—the air velocity I mean—*V*, and the velocity of wind* v*. If the distance to N-base is L, then the flying time against the wind will evidently*

be L /(V — v) *and the flying time with the wind* L /(V + v). *The time of the round trip is therefore:*

$$L/(V-v) + L/(V+v) = 2LV/(V^2 - v^2) = 2L/V\,(1/1-(v^2/V^2))$$

Since 2L/V *would be the total flying time without any wind, it follows that on a windy day the flying time will always be longer. If, for example, the wind velocity were half that of the plane* (v/V — 1/2), *the flying time would be lengthened by a factor* 1/(1 — ¼) = 4/3. *If the wind velocity were only a. little bit less than that of the plane, it would take days to fly as short a distance as that against the wind, and if* V *equals* v *the flying time would become infinite. Of course, with the jet I am flying, it doesn't matter much, and I gave that reason for postponing my night only to puzzle you. Actually they phoned me to say that the man I wanted to see there will not arrive until tomorrow."*

"All right, if you're not going, tell me about Michelson's experiment."

"Well, it was this way. In order to account for the way light travels in space, some scientists supposed that there was something they called ether that fills all space in the universe. Michelson figured that if this ether existed, we should be able to detect the ether wind blowing past us as earth moves through space. The earth revolves around the sun at the rate of about six miles a second, so we should notice it almost the same way you feel the air wind in your face if you fly in an open cockpit.

"In his experiment Michelson sent two beams of light, one in the direction of the hypothetical ether wind, and one in the direction perpendicular to it. At the ends of their tracks both light beams were reflected back to the source and compared upon arrival. The beam traveling with and against the ether wind was expected to be delayed in the same way that my plane would be if I had flown to N-base today, while the beam traveling in a perpendicular direction was expected to be delayed a different amount of time. Comparing the time of arrival of both beams, Michelson hoped to detect the motion of our earth through the light ether."

"Well, did he?"

"No. It turned out that both beams returned to the point of origin simultaneously, without any delay whatsoever. This puzzled the physicists for quite a while, until Einstein explained the negative result by revolutionizing the old ideas of space and time in his famous theory of relativity. But I never thought my trip to N-base was going to get me into Einstein. It's funny how that problem of flying with and against the wind can fool you. The first impression is bound to be that the effect of wind cancels out"

HOMING MISSILES

"If you are so clever with that kind of problem," said the first officer, "I can give you a very knotty one I recently heard from a friend. You have four homing missiles located originally in the vertices of a square twenty by twenty miles in size. Each missile homes on another missile, as you can see from this drawing." He made a quick sketch. "And the velocity of the missiles is, say, one mile per second. As a result of this arrangement, each of the four missiles will gradually turn to the right, always keeping the correspond-ing target missile on the nose, and they will ultimately collide in the middle of the square. The problem is to find out how long it will take for the missiles to collide."

"That's quite a complicated mathematical problem," said Jack thoughtfully. "It probably should be treated in the polar-coordinate system with the origin in the center of the square, and I am sure I can write the differential equation for the trajectory. But of course I can't do it right off."

"That's just the point. Everybody who knows higher mathematics can presumably solve that problem, but the trick is to find the solution which is so simple that anybody could understand it."

"I don't think that can be done in this case. The trajec-tories of the missiles are probably rather tricky geometrical curves."

"The trick is that you don't even need to know what the trajectories are. All that you have to consider is the fact that, all during the flight, the four missiles are located in vertices of a square which shrinks and rotates clock-wise. Forget about the rotation, and think about the shrinkage only. Since the missiles home at each other, the velocity of each one is always directed along one side of the shrinking square toward the missile located at the next vertex. Thus, the rate at which the sides of the square shrink is equal to the velocity of the missiles, that is, one mile per second. And since the missiles were originally twenty miles apart, it will take twenty seconds for them to collide."

"Very, very interesting," said Jack. "We should try it some time flying four fighters on maneuvers. It would be quite a lot of fun."

REFUELING

"Well, here's another problem which may interest you fellows," said another pilot. "Suppose you have to deliver a bomb in some distant point of the globe, the distance being much greater than the range of the plane you are going to use. Thus, you have to use the technique of refueling in the air. Starting with several identical planes which refuel one another, and gradually drop out of the flight until the single plane carrying the bomb reaches the target, how would you plan the refueling program, and how many planes will you need to carry out the operation? We will assume for sim-plicity that airplane fuel consumption can be measured in miles per gallon and is independent of load."

"Oh, just tell us," said one of the pilots. "We are all too tired to work out these problems."

"Very well, suppose you originally have D *identical planes including one which carries the bomb, and the gas tanks of all the planes are filled up at the start. As the flight proceeds there comes a moment when the amount of gas left in any one of the* n *planes is just enough to refuel completely-all the other* (n-1) *planes. Thus, for example, if there were originally ten planes each carrying ten thousand gallons of gas, they fly until nine thousand gallons is left in each. At this point the entire gas supply of one of the planes* is *used to refuel the remaining planes, the emptied plane falls off, and the other nine proceed with full gas supply. The next operation is carried out when the fuel supply of each has been reduced by one-ninth, one of the planes lands, and the other eight proceed completely refueled. The subsequent refueling take place when one-eighth,* one-seventh, *etc., of the fuel is used up, until only one plane is left, and it reaches the target using up the last drop of gasoline in its tanks,*

"Denoting the range of the planes by R *and the number of planes by* n, *we find that the total distance which can be covered is:*

$$D = R/n + R/n\text{-}1 + R/n\text{-}2 + \ldots\ldots + R/3 + R/2 + R =$$

$$R\, [\, 1 + ½ + 1/3 + \ldots + 1/(n - 2) + 1/(n - 1) + 1/n \,]$$

Thus, *for example, if* n *equals* 10, *the sum in the brackets equals* 2.929 *which means that by using this refueling method one can reach the target at a distance almost three times the range of the individual planes."(The* reader will notice that this problem reduces to the same mathematical series as the problem of the piled-up dominoes)

Epilogue:

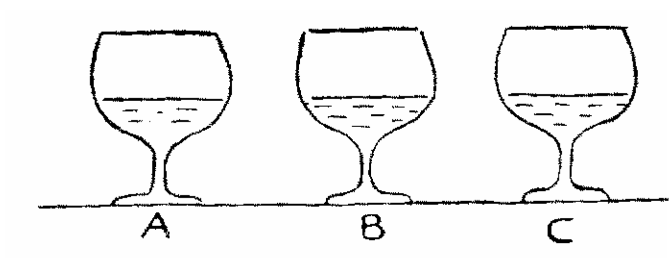
The Moral

The book was finished. Three of us, Gamow, Stern, and another puzzle fan, Theodore von Karman—were sitting at the Landfall in Woods Hole, dividing a bottle of Slivovitz. The question arose as to whether the brandy was being divided evenly.

"Assuming that we are all very selfish individuals, could we divide the brandy in such a manner that each of us is satisfied that he is getting at least as much as anyone else?" Gamow asked. "We are all familiar with the way this problem is solved in the case of two squabbling children. One child divides the booty into what he considers to be two equal shares so that he will be content with either one. 'The other child is then given the privilege of choosing the portion that he prefers. Could this be extended to the case of three people?"

Von Karman smiled and turned to Stern. "Allow me to rephrase the problem slightly, and then I am sure that you should be able to solve it. Consider the problem to be re-worded to state that each of us must be satisfied that he is getting at least his fair share of the total (at least one-third). Now you should be able to do it."

"Yes, I think I see how to do this," said Stern. "Von Karman divides the brandy into what he considers to be three equal shares, so that he will be content with any one.



"Now if Gamow thinks that A holds the most and I think that B holds the most, then there is no problem. Gamow gets A, I get B, and von Karman gets C. A problem can arise only if both Gamow and I think that A holds the most. Even in this case, if we both agree that B has the second largest volume, the problem is reduced considerably. Gamow and I merely have to divide A and E between us in the same manner as the two children, and von Karman is left with C.

"The only real difficulty occurs when Gamow and I both agree that A has the most, Gamow thinks that B has the second most, and I think that C has the second most. In this case again I let Gamow divide A and B into what he considers to be equal portions. In doing so, he pours from A into B.

"Now if, after this operation, I still think that A has the most and take that, then Gamow must take B since he had originally thought that C had the least, and therefore definitely less than one-third of the total. Of course von Karman would prefer B, since after he had divided into what he thought were equal portions; B had more brandy poured into it. But since the problem stipulated that each of us merely had to be satisfied that he was getting at least his fair share, von Karman would have to be content even when Gamow chose B.

"If, after Gamow's pouring from A into B_r I were to take B, then .by the same argument Gamow would be content "with A, and von Karman would again be content with C.

"If, after the pouring, I decide to take C, then von Karman is allowed to take B, and Gamow is content with A."

Von Karman smiled while listening to the solution and then remarked, "Now do you see why I changed the stipulation of the problem from a case in which each of us was to be satisfied that he was getting at least as much as anybody else to one in which each of us merely had to be content that he was getting at least his fair share? One of the steps in your solution shows that the problem as originally stated is unsolvable.

"We will classify this as the riddle with a moral in human relations"

End