ROMPING IN NUMBERLAND

P.K. Srinivasan

Preface

Children are curious by nature. They look at everything with wondrous eyes. They are eager to find out relations and connections not only among concrete things but also between abstract ideas according to their level of maturity and capability.

Pattern finding is inherent in human mind. Given opportunity and encouragement, a child revels in it and blossoms into an explorer Of digital form. The system of natural numbers abounds in wonderful patterns, simple to complex, trivial to deep. Children get introduced to natural numbers in their primary schooling. They learn to do addition and subtraction, multiplication and division with the numbers.

This book attempts to show how children can enter the threshold of the fascinating world of number theory just by using these fundamental skills and discovering for themselves through the exercise of their intuitive powers many of the simple but beautiful number properties. This provides the children joy and self-confidence without. which they cannot become motivated learners in mathematics.

It is hoped that imaginative parents and interested teachers will welcome and patronize this novel venture and share with the author their experiences.

Suggestions for improving subsequent editions will be duly acknowledged.

P. K. SRINIVASAN

1. JOIN ME

I am in class VII. I love reading stories and playing games.

My uncle is a Professor of Mathematics. He would often tell me, ‘You too can make some simple discoveries and begin to enjoy yourself in mathematics, if only you care to know the various kind of natural numbers, identify them without hesitation and play with them. What you need to use are simply the basic skills of addition and subtraction, multiplication and division which you have already acquired in lower classes’. For sometime, I did not take his suggestion seriously.

One day my uncle was invited to speak to the members of a Middle School Mathematics Club named after Euler, one of the greatest mathematicians of all time. I went along with my uncle to hear him speak to youngsters like me and to see for myself how they responded to him.

My uncle talked on how to become a junior mathematician and have an exciting time in the following words:

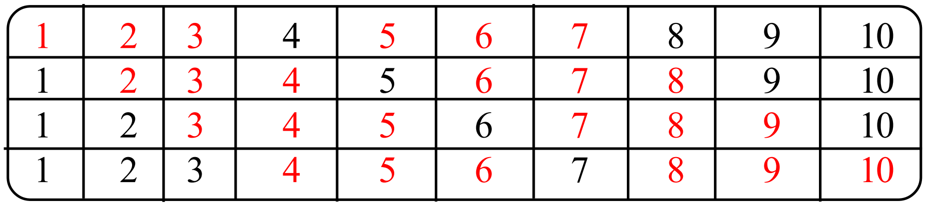
These are days of fantastic discoveries. Discoverers are in great demand all over the world in all walks of life. One cannot become a discoverer all of a sudden. One should cultivate a taste for discovery from one's childhood.

Numbers form the ideal ground for making discoveries as it does not involve any expense besides being highly rewarding. You may wonder how it can be so. Once a person acquires the basic skills of addition, subtraction, multiplication and division, he can be sure of exploring for himself many beauties in the behavior of natural numbers.

Since all of you know how to add, subtract, multiply and divide, you have the readiness for making discoveries. The more you try discovering, the more you turn out to be an explorer.

Pattern finding is within the reach of you all and that forms, by and large, the key for discoveries in the wonder- land of numbers. You may even call it the wonder sea of numbers, if you like. Pattern finding leads you to see relationships among numbers and through relationships of numbers, you can spell out some beautiful properties or some enchanting peculiarities of numbers.

Well, I have brought for you some Visualisation Charts. Here is one.



1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10

You may wonder why the sequence of natural numbers is repeated in this chart. It is just to help you visualise all the sets of three consecutive natural numbers.

Take set by set, find the sums or products of all or some of the numbers in them, study what you get or compare what you get with the rest of the numbers to discover the pattern or patterns. Sometimes the pattern may be on the surface; sometimes the pattern may be deep.

What does not hold good in all cases will be a peculiarity. Don't waste your time on peculiarities, to start with. Look for properties that seem to hold good in all cases.

For instance, take the three consecutive numbers, set by set, starting with 1, 2, and 3. 1 + 3 is 2 x 2. This is true. But 2 + 4 is not 3 x 3. So to say that the sum of the end numbers in a triad of consecutive numbers is the square of the middle number does not hold good for all triads. So this is a peculiarity and not a property of three consecutive natural numbers.

On the other hand, consider

1 + 3 = 2 x 2
2 + 4 = 2 x 3
3 + 5 = 2 x 4 and so on.

You can go on finding that the sum of the end numbers in any triad of consecutive natural numbers is double the middle number. Hence this is an example of property.

It is just like this. Suppose your headmaster gives you a certificate which says that you are sometimes intelligent. Will you be happy to receive it? Don't you want the certificate to say simply that you are intelligent, meaning thereby that you are intelligent always!

So you should be on the look out for properties hidden among the numbers. If you come across peculiarities, make a note of them and just forget about them for the present. You may treat them as curies or freaks just as you do when you see a person with more than five fingers in a hand.

If a certain pattern repeats itself in a number of cases, you can smell the existence of a property. Remember it is only a guess, or to use mathematician's language, a conjecture. It cannot be accepted as true till it is proved. The strength of a proof of a property does not lie on the number of particular cases or examples wherein the property holds good. Proving is indeed more exciting. You can take it later. We shall start making guesses. The only thing which we should consider is that each guess we make should be plausible and it should be based on the truth of a number of cases.

Well, I would like to see now how many of you are potential junior mathematicians. Make use of the Visualisation Chart before you. You have seen that the sum of the end· numbers in every set of three consecutive natural numbers is double the middle number. Look for more.

Members of the audience were seen writing and whispering. Some of them raised their hands up. I had also spotted a pattern by finding the sum of the numbers in each triad.

1 + 2 + 3 = 6
2 + 3 + 4 = 9
3 + 4 + 5 = 12
4 + 5 + 6 = 15 and so on.

I jotted down the pattern in a slip of paper with the note that the sum of three consecutive numbers is a multiple of three and passed it on to my uncle. He looked at it and gave his smile of approval. T felt proud.

He asked some of the members who put up their hand to come out and present their discoveries. To my great joy, I found some of them having done the same thing as mine. Some others had considered the products of three consecutive numbers and wrote on the blackboard.

1 x 2 x 3 = 6
2 x 3 x 4 = 24
3 x 4 x 5 = 60
4 x 5 x 6 = 120 and so on.

The discoverers were asked to give the property in words. What they said ran as follows:

The product of three consecutive numbers is a multiple of six. Six is a factor of the product of any three consecutive natural numbers. Multiply any three consecutive numbers and divide the answer by six; you find an exact division; and so on, all meaning almost the same thing.

My uncle was quite happy with the enthusiastic response from the youngsters. While he was congratulating them and commending their discoveries, three hands were seen going up from among the members to his left. When he looked that side, three students stood up and said that they had discovered another property and it was much more interesting. They were invited to come before the audience and write simultaneously on the board the pat- terns found by them. What they wrote ran as follows:

1 x 3 = 2 x 2 - 1
2 x 4 = 3 x 3 - 1
3 x 5 = 4 x 4 - 1
4 x 6 = 5 x 5 - 1 and so on.

1 x 3 + 1 = 2 x 2
2 x 4 + 1 = 3 x 3
3 x 5 + 1 = 4 x 4
4 x 6 + 1 = 5 x 5 and so on.

1 x 3 is 1 less than 2 x 2
2 x 4 is 1 less than 3 x 3
3 x 5 is 1 less than 4 x 4
4 x 6 is 1 less than 5 x 5 and so on.

One of them was chosen to state the property in words. He said:

'When you multiply the first and the last of three consecutive numbers, your answer is one less than the square of the middle number.'

I became envious of them, as I did not discover it for myself.

My uncle patted them on their backs and asked the audience, 'Which of the two properties so far discovered by you is deeper?' Almost all the members shouted, 'The second one, the second one. My uncle observed, 'The' deeper is the property discovered, the greater is the fascination of discovery.' Thus he set the tone for the climate of discovery.

Then he said that they should know and be able to identify the kinds of natural number in order to make more discoveries. He asked them to give examples of even and odd numbers, multiples and factors, prime and composite numbers, squares and cubes etc, he explained how to describe them and determine them. He introduced the vocabulary consisting of successor, predecessor, corresponding, to facilitate communication. He also told them that they should be able to list numbers of each kind upto 100 and in the case of cubes upto 1000.

While he was talking about these, a hand went up. He stopped his talk and asked the youngster if he had discovered something deeper still. He said he had found a peculiarity.

1 + 2 + 3 = 1 x 2 x 3
But 2 + 3 + 4 (is not equal to) 2 x 3 x 4

After congratulating him, my uncle observed that a peculiarity could become the starting point for finding the set of numbers with the same peculiarity. The youngster said there would be none. My uncle asked him to ignore the conditions that they should be triads and they should be consecutive numbers and see if there could be numbers having the 'peculiarity of having the sum of their proper factors equal to the number.' The youngster said it would take a lot of time and he was eager to get them from the lecturer himself. My uncle asked him to consider 1, 2, 4 ,7 and 14. The youngster worked out their sum and found it to have the 'peculiarity.' He said, he would like to find more for himself.

My uncle commended his attitude and went on. 'If you collect more such numbers, the-peculiarity become the property of those numbers. How wonderful!' I enjoyed his observation. So did the audience. One youngster asked: 'What are such numbers?' My uncle explained thus:

$1 + 2 + 3 = 6$, $1 + 2 + 4 + 7 + 14 = 28$. Observe that 1, 2, 3 are factors; proper factors to be precise, of 6 and 1, 2, 4, 7, 14 are proper factors of 28. Numbers such as 6, 28 have the property that the sum of the proper factors of each is the same as the number itself. One youngster asked if such numbers had a name. 'Yes, they are called perfect numbers,' replied my uncle.

He concluded his talk by congratulating warmly all those who came out with discoveries made on the spot and everybody for the great interest and enthusiasm shown. 'With a little patience and sustained interest,' he said, 'everyone of you can blossom into a junior mathematician. He presented the club with a set of Visualisation Chart he had brought with him.

The meeting was over. On the way home, I told my uncle that I was envious of those who gave deeper results. I told him that I would visit him every weekend for making discoveries in his presence. My uncle was quite pleased to hear this and said, 'you are quite welcome. I would certainly allot sometime for you to spend with me'.

Thus my romping in numberland started. I often quote verbatim the discussion between me and my uncle so that you can also benefit from my exploratory. Will you join me?

2. KINDS OF NUMBERS

This was my first discovery weekend with my uncle. He wanted me to brush up my familiarity with the kinds of natural numbers so that I could identify them as they showed up.

Even and Odd

He picked up a collection of marbles and asked me, 'Find out without counting if the number of the collection is even or odd.' I did not know how to do it. I could count the marbles and say if the number was even or odd by considering the digit in the units place of the number.

He suggested that I should put the marbles in the collection in pairs and see what happened. Once I started doing so, I found that the number of the collection was even, as all the marbles in it could be put in exact pairs. He gave me another collection When I put the marbles in it in pairs, one marble remained unpaired and so the number of the collection was odd. I asked him,’ Why is it not enough to count and examine the digit in the units place of the number got and say the number even when the digit is a multiple of 2 and odd when the digit is not a multiple of 2?’

He appreciated my question and asked me to recall to what base I counted. I said ‘To base ten’. He then wanted me to give twelve to base seven. I said, ‘one five (15) to base seven’. At once I could see why he wanted me to under- stand evenness of a number by pairing. Fairing is independent of base. ‘On the other hand if we say that a number is even when it is exactly divisible by two and it is odd when it is not exactly divisible by two, we are correct’, my uncle observed.

	2		4		6		8		10		12		14		16		18		20
1		3		5		7		9		11		13		15		17		19	
	22		24		26		28		30		32		34		36		38		40
21		23		25		27		29		31		33		35		37		39	
	42		44		46		48		50		52		54		56		58		60
41		43		45		47		49		51		53		55		57		59	
	62		64		66		68		70		72		74		76		78		80
61		63		65		67		69		71		73		75		77		79	
	82		84		86		88		90		92		94		96		98		100
81		83		85		87		89		91		93		95		97		99	

Finally he asked me to list all the even and odd numbers upto 100 by putting them in two tiers. It was not at all difficult.

The list shows that in base ten system, the even numbers end in

0, 2, 4, 6 and 8 and

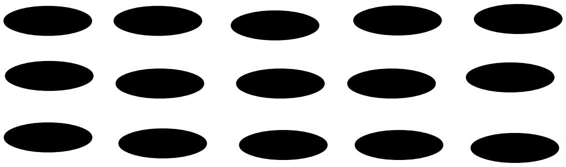
the odd numbers in

1, 3, 5, 7 and 9.

Multiples and Factors

Next he asked me, ‘Can there be any collection, the objects of which cannot be arranged in a single row (or column)?’ I could say easily, ‘there can be none’. He then asked me, ‘What does it mean in terms of divisibility?’ hesitatingly I said, ‘1 is a factor of any natural number and any natural number is a factor of any natural number and any natural number is a factor of itself.’ He asked me to rephrase what I said using multiple instead of factor. I said. ‘Any natural number is a multiple of 1 and any natural number is a multiple of itself.

He asked me to use the bottle tops in the bowl in the cupboard there and set up a rectangular array. I could not get him. He explained that they should be placed in more than one row (or column) and the same number of them should be in each row (or column). I set one like this:



‘Count the bottle tops used in your arrangement.’

‘Fifteen.’

‘How many rows are there?’

‘Three.’

‘How many are there in each row?’

‘Five.’

‘Don’t you see that five is repeated three times and that gets you fifteen?’

‘Oh! I see. So 15 is the multiple here and 5 and 3 are its factors.’

‘That is it: 15 is a multiple of 5 and 3.

5 or 3 is a factor of 15. If a number can be expressed as a product of numbers, the product number is a multiple and numbers forming the product are its factors.’

‘Is an even number a multiple?’

‘Yes. It is a multiple of 2.’

‘What about an odd number?’

‘Well, 21 is a multiple of 7.’

‘In fact, every number is at least a multiple of 1 and every number is at least a factor of itself. Right?’

‘Yes, I get the point’

‘Give a common factor of an even number and an odd number. ‘

‘How can I give that when I don’t know them?’

‘Why not?’

‘I get it. It is 1. Am I right?

‘Yes, you are. An even number and an odd number must have at least one common factor. What can that factor be?’

‘That can only be 1’

‘Can you give me one such example of an even number and an odd number?’

‘Well, 2, 3.’

‘Another example?’

‘3, 4.’

‘Do you mean to say that they should be consecutive?’ ‘Not necessarily. It is easy to give that way. That is all.’

‘That is good. We shall meet next weekend;

I took home some bottle tops with me for playing with them.

Prime and Composite Numbers

During the week, I set up many rectangular arrays and discovered that it was not possible to arrange bottle tops in every collection in a rectangular array. Of course I could place any collection in a row (or column).

When I met my uncle at the weekend, I told him about my experience. ‘There are collections whose objects cannot be arranged in a rectangular array.’

‘Yes. It is true. The numbers of such collections are called non-rectangular. However, objects of any such collection can be arranged in a single row. What does it show?’

‘It shows that a non-rectangular number has got only two factors. For instance 7 is non-rectangular and it has only two factors which are 1 and 7.’

‘That is good. Non-rectangular numbers are the so called prime numbers and rectangular number composite numbers.’

‘Oh! I understand. A prime number has only two factors and a composite number has more than two factors. Am I right? What about 2 and 1?’

‘You cannot answer your question by rectangular arrangement. But you can apply number of factors tests.’

‘I get it. Two has only two factors 2 and 1. So 2 is prime. Rut 1 has only one factor 1. So it is neither prime nor composite. Am I right?’

‘You are perfectly right. Would you like to have a number quiz?’

‘I would love to.’

‘What is the first prime number?’

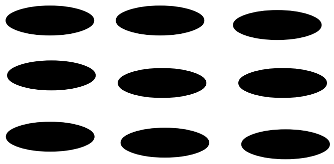
‘Two.’

‘Correct. What is the first odd prime number?’

‘Three.’

2	3		5		7				11		13				17		19				23			
		4		6		8	9	10		12		14	15	16		18		20	21	22		24	25	26
		29		31						37				41		43				47				
27	28		30		32	33	34	35	36		38	39	40		42		44	45	46		48	49	50	51
	53						59		61						67				71		73			
52		54	55	56	57	58		60		62	63	64	65	66		68	69	70		72		74	75	76
		79				83						89								97				
77	78		80	81	82		84	85	86	87	88		90	91	92	93	94	95	96		98	99	100	-

‘Yes. I have. Let us take for instance 9 objects. The number of rows is 3 and the number of objects in a row is also 3.’



‘Such numbers arise from a special kind of rectangular arrays. What can they be called?’

‘Square?’

‘Good. As they arise from square arrays, they ;re square numbers. Are rectangular numbers square?’

‘Only when the number of rows is the same as the number of objects in a row.’

‘Then, is every square number rectangular?’

‘Yes. It is.’

‘So every square number is rectangular, but not the other way. What about the factors of a square number?’

‘9 = 3 x 3, 4 = 2 x 2, and so on. A square number has two equal factors. Right?’

‘Yes, you are. Now, is 1 a square number?’

‘How should we find it out as we cannot have it in a square array?’

‘A square number has two equal factors. In other words, it is the product of two equal factors. But 1 has more than two equal factors as we can write 1 as 1 x 1 x Ix etc.’

‘Be careful. Don’t get confused. Can’t you write 1 as a product of two equal fact6rs?’

‘Oh I see. 1 is 1 x l. So 1 is a square number. Right?’

‘Yes. You are. Now list the natural numbers in two tiers showing respectively square and non-square numbers. ‘You may see the table on the next page:

1			4					9							16									
	2	3		5	6	7	8		10	11	12	13	14	15		17	18	19	20	21	22	23	24	
										36													49	
26	27	28	29	30	31	32	33	34	35		37	38	39	40	41	42	43	44	45	46	47	48		50
													64											
51	52	53	54	55	56	57	58	59	60	61	62	63		65	66	67	68	69	70	71	72	73	74	75
					81																			100
76	77	78	79	80		82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	

I enjoyed listing the natural numbers in two tiers so as to show up squares and non-squares, as there is a pattern in the number of non-squares appearing between two consecutive squares.

'You can call them 'interval sequence' if you like.

'O.K. The numbers of non-squares in interval sequences are seen to be multiples of 2.'

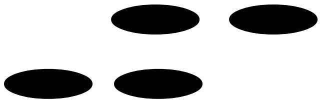
'What an interesting- pattern!'

'Yes. Indeed.

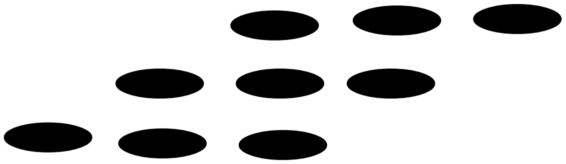
Cubes and Non-Cubes

Next my uncle passed on to cube numbers. He placed before me a collection of bottle tops and wanted me to arrange them in such a way that the number of layers coming one over the other was the same as the number of rows and the number of rows the same as the number of bottle tops in each row.

I succeeded in doing it easily. The first one consisted of two layers, two rows in each layer and two bottle tops in each row.



'Suppose there were to be three layers for a cube number to be built. How will you build it?'



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'I get it. There should be three layers, three rows in each layer and three bottle tops in each row. Now I can build it.'

'That is good. A square number is a product of two equal factors. How many equal factors would go into a product to give a cube number?'

'8 = 2 x 2 x 2, 27 = 3 x 3 x 3. A cube number is a product of three equal factors.'

'Good. What about 1? You know it is a square number. Can it be a cube number too?'

'Since 1 can be shown as 1 x 1 x 1, 1 can be taken as a cube number. Am I right?'

'Indeed you are. If a number can be expressed as a product of three equal factors then that number is a cube. Now 8 is the cube of 2. Do you know how 2 is related to 8?'

'2 x 2 = 4. 4 is the square of 2 and 2 is the square root of 4. Likewise since 2 x 2 x 2 = 8, 2 is the cube root of 8. Am I right?'

'Yes, you are. Now list the natural numbers in two tiers showing respectively cube and non-cube numbers upto 100.

As in the case of listing squares and non-squares, I found listing cubes and non-cubes interesting. The table is given on the next page.

1							8																	
	2	3	4	5	6	7		9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	27																							
26		28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
													64											
51	52	53	54	55	56	57	58	59	60	61	62	63		65	66	67	68	69	70	71	72	73	74	75
76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	99	99	100

The interval sequences are longer. The numbers of non- cubes in interval sequences are 6, 18, 36, 663, 90 and so on and these are all multiples of 6.

‘O.K. List only cubes upto 1000. How can you do it?’

‘By taking products of three equal factors, the factors being 1,2,3 etc. I can do it.’

‘Good. List the cubes now.

‘The cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000

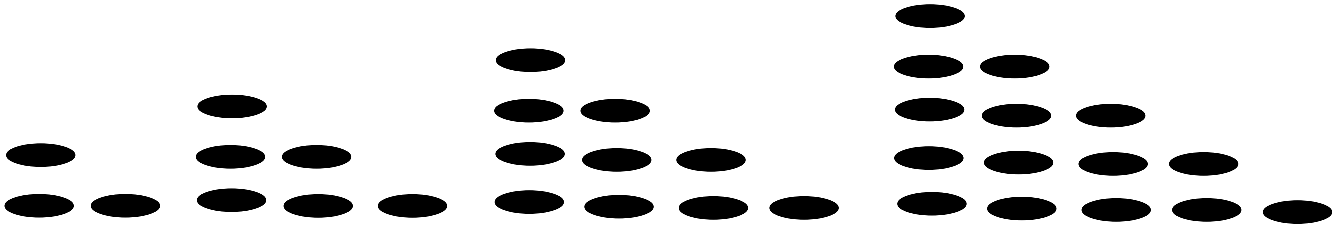
‘Well. We shall take up triangular numbers next weekend.

‘What are triangular numbers?’

‘Hold yourself in patience till we meet next week.’

Triangular and Non-triangular Numbers

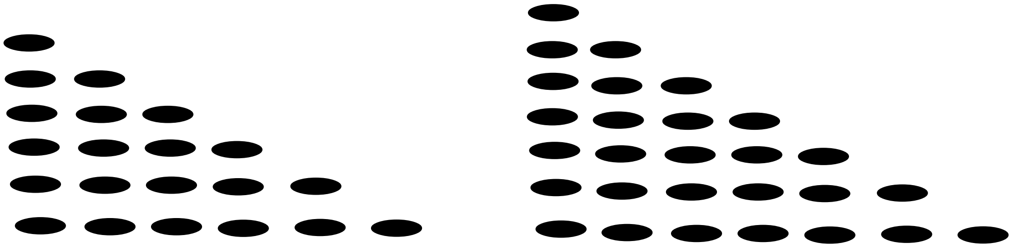
He wanted me to watch what he was doing. He took tops and made the following arrangements in sequence.



And wanted me to make two more arrangements that would follow in the sequence.

First I observed the arrangements carefully.

‘Rows are not made up of equal number of objects and each row in any arrangement has one more object than the previous row.’ Having observed this pattern, I set up the next two arrangements in the sequence as follows:



‘What shape does each arrangement take?’

‘Triangular. ‘

‘Yes, these represent triangular numbers. Can you count and list the triangular numbers so far go?’

‘Oh, I see. 31 6, 10, 15, 21, 28.

‘Without building triangular arrangements, can you give the next two triangular numbers?’

‘Add 3 to 3, you get 6. Add 4 to 6, you get 10. Add 5 to 10, you get 15. Oh! I see. ’The next two triangular numbers to follow 28 are 28 + 8 = 36 and 36 + 9 = 45. Am I right?’

‘Right. Is 1 triangular?

‘1 is a square number. 1 is a cube number.

Now the question is ‘Is 1 triangular?’ Well, let me go backwards and see:

15 - 5

10 - 4

So the next one should be 3 - 2 1· So 1 is triangular? Am I correct?’

‘That is really wonderful. The way you have gone about is excellent. ‘Well, can you list the natural numbers.

‘In two tiers showing respectively triangular and non- triangular numbers upto 100?’

‘Now-a-days you see your way fast. Come on. Finish listing now.’

I did not find it difficult. For after a certain stage, I found the numbers of non-triangular numbers in the interval sequences to be simply the consecutive natural numbers starting from 1.

Table is given on the next page.

1		3			6				10					15						21				
	2		4	5		7	8	9		11	12	13	14		16	17	18	19	20		22	23	24	25
		28								36									45					
26	27		29	30	31	32	33	34	35		37	38	39	40	41	42	43	44		46	47	48	49	50
				55											66									
51	52	53	54		56	57	58	59	60	61	62	63	64	65		67	68	69	70	71	72	73	74	75
		78													91									
76	77		79	80	81	82	83	84	85	86	87	88	89	90		92	93	94	95	96	97	98	99	100

‘Would you like to have some quiz?’

‘Why not?’

‘What is the number next to one which is both triangular and square?’ I went through the list and said,’36.

‘Good. It is correct. How many triangular numbers are there between 1 and 100?’

‘Twelve’

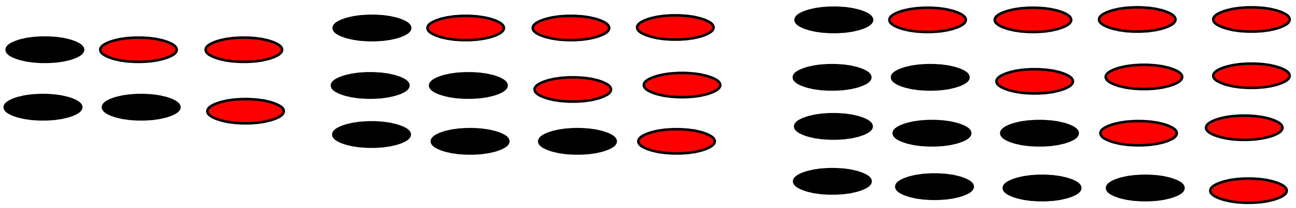
‘That is right. Now the first triangular number is 1, the second 3, the third 6, the fourth 10 and so an. Suppose I ask you to give,say,20th triangular number. How will you find it?’

‘Let me see. Let me form a table.

S. No.	Triangular No.
1	1
2	3
3	6
4	10 and so on.

‘I think I should add progressively that is,. 2 to 1 to get 3, 3 to 3 to get 6, 6 to 4 to get 10, 10 to 5 to get 15, 15 to 6 to get 21 and so on till I reach the 10th triangular number. Is there not a shorter way?’

He asked me to place upside down another equal triangular number and observe the numbers of the composite arrangement. I did as shown overleaf.



‘I get rectangular numbers.’

‘Do you see anything special about these?

Compare the number of rows and the number in each row.’

‘Oh! I see. In these rectangular numbers the number of objects in each row and the number of rows are consecutive natural numbers. Do they have a special name?’

‘Well, they are called oblong numbers.’

‘Can you list them?’

‘Are they not simply twice triangular numbers? So they are 6, 12, 20 and so on.’

‘Yes, you are very correct.’ ‘I get it now. A triangular number is half the product of two consecutive numbers.

1st triangular number is (1 x 2)/2

2nd triangular number is (2 x 3)/2

So the 10th triangular number is (10 x 11)/2 and so on.
Right?'

'Yes. You have got it. Now what is the first natural number which is a square, a cube and a triangular number as well?'

'It is 1. Is it not?'

'Very good. It is indeed.'

'Now consider the numbers of non-cubes in interval sequences. 'You mean 6, 18, 36, 60 and 90 and so?' 'Yes, that is it. You know they are multiples of 6. Now identify what multiples of 6 give respectively 6, 18, 36, 60, 90 and so on" 'You mean 1, 3, 6, 10, 15. Oh! I see the multiples are simply the triangular numbers.'

Successor and Predecessor

Now you need to know the use of a few more words to help easy communication. Can you give the natural numbers in sequence?'

'1, 2, 3, 4, 5, 6 and so on.'

'Which number has no predecessor, I mean the number coming immediately before it?

'Which number has successor, I mean the number coming immediately after it?'

'Every number in the sequence.'

'Which number has no successor?'

'None.'

'How do you know it?'

'When I am given any number in the sequence I can give its successor by adding 1 to it.'

'Good. Now give the whole numbers in sequence.'

'0, 1, 2, 3, 4, 5 and so on:

'Has 1 its predecessor in this sequence?'

'Yes, the predecessor of 1 is 0.'

'Can we say that a number and its successor or a number and its predecessor are consecutive numbers?'

'Yes, we can say so.'

'Now list the natural numbers in a row then list the even numbers in the second row so as to see these numbers match with the numbers in the first row.'

1	2	3	4	5	6	7	8	9	10….
2	4	6	8	10	12	14	16	18	20….

'Good. How can you get the second row from the first row?'

'By multiplying each number by 2.'

'So the even number corresponding to 1 is 2. The even number corresponding to 2 is 4.'

‘What is the even number corresponding to 7?

‘Fourteen. ‘

‘Good. What is the natural number corresponding to the even number 18?’

‘Nine.’

My uncle congratulated me on my successes so far and told me that from the following weekend, I would have a much more exciting time through my romping rounds in Numberland.

‘Why not number sea, Uncle?’

‘Have it that way if you like.

3. WARMING UP

I met my uncle this time with great expectations. He reminded me that it was enough if I relied on my skills of adding and subtracting, multiplying and dividing, squaring and cubing in my ventures now. He advised me to be watchful about the emerging number patterns or relations. He assured me that my background knowledge about the kinds of numbers would be of great help in spelling out my discoveries.

He showed me some flash- cards. The first one had this on it.

A	**B**	**C**	**D**
3	**4**	**7**	**13**
8	**1**	**9**	**9**
5	**7**	**12**	**36**
9	**2**	**-**	**-**

‘Examine the table. Find out first how C numbers are related to A and B numbers and fill up the blank in the last row with the appropriate C number.

‘I see the pattern, as it is on the surface.’

$$3 + 4 = 7$$
$$8 + 1 = 9$$
$$5 + 7 = 12$$

So C number for the last row is 9 + 2 =11.’

‘Can we say that C number is the sum of A and B numbers?’

‘Good. We can. We can even put it this way C = A+B. Now find out how D numbers are related to A and B numbers and fill up the blank in the last row with the appropriate D Number.’

‘The pattern is not on the surface. Let me try different ways. I have already added. D number do not seem to be related to the corresponding sums of A and B numbers. Subtraction will obviously be out of place. Let me form products of A and B numbers.

Will that work?’

‘Draw up a table of product numbers and the corresponding D numbers and see.’

‘Here is the table

A x B	D
12	13
8	9
35	36

Oh! I see. Comparing the product numbers with their corresponding D numbers, I find the D numbers are successors. That is to say, D numbers are the successors of A x B numbers. So the D number for the last row is 9 x 2 + 1, that is 19, Right?’

‘Very good. Can we say that D = A x B + 1?’

‘Oh! yes. I wanted to say it myself. It is so’

‘Now take this flash card showing

P	**Q**	**R**
3	**8**	**10**
5	**24**	**26**
6	**35**	**37**
4	**-**	**- and**

Discover first how Q numbers are related to P numbers and then how R numbers are related to P numbers and find the missing ones in the last row.

‘Am I to add a P number to itself or multiply a F number by itself before looking for the pattern?’

‘Well, you can try that way. Don’t forget drawing up a table.

‘Let me add and draw up the table”

P + P	**Q**
6	8
10	24
11	35

I don’t see any pattern. Let me draw up the table that would be got by multiplying P itself:

P x P	**Q**
9	8
25	24
36	35

Now the pattern appears. Q number is the predecessor of the square of P number. So the Q number corresponding to P number 4 is 16-1, that is, 15.

My uncle was watching my work and appreciated me warmly when he saw me writing 15.

‘Now you can easily find the relation between Ii and P numbers.’

I looked at the squares of P numbers and R numbers. The pattern was there.

‘R numbers are simply the successors of the squares of P numbers and so the R numbers corresponding to the P number is 16 + 1, that is 17.

My uncle was visibly pleased with my achievement. He said I was quite ready for the discovery spree to commence from the next weekend.

4. ROMPING ROUND 1

Pairs of Consecutive Natural Numbers

I saw on my uncle’s table a stack of Visualisation Charts, like the one he had used in the math club meeting.

He asked me to pick up the Pink Visualisation Chart and examine it.

He said, ‘you are to study the pairs of consecutive numbers showing up’

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10

‘Can you try?’

‘I am able to see the pairs of consecutive natural numbers.’

1, 2
2, 3
3, 4
4, 5 and so on.

Let me find their sums and see.

1 + 2 = 3   4 + 5 = 9
2 + 3 = 5   5 + 6 = 11
3 + 4 = 7   6 + 7 = 13   and so on.

3, 5, 7, 9, 11, 13 etc, are all odd numbers.

I get it. The sum of any pair of consecutive numbers is an odd number. Right?’

‘A good shot. Can you find any other property?’

‘Addition has given a property of pairs of consecutive numbers. Why not I do subtraction and see?

2 - 1 = 1
3 - 2 = 1 and so on.

The difference of two consecutive numbers is one

‘Do you think it is so nice as the one you have got by addition?’

‘No. It is not so nice.’

‘We call it a trivial property.’

‘Next let me try multiplication:

1 x 2 = 2 4 x 5 = 20
2 x 3 = 6 5 x 6 = 30
3 x 4 = 12 6 x 7 = 42 and so on.

2, 6, 12, 30, 42, etc. are all even. So, the product of two consecutive numbers is even.’

‘Another result as goods as the one got by sums of pairs of natural numbers.

‘Can I get a deeper result?’

‘If you get one, that would be wonderful.’

‘Suppose I take the square of the first number in a pair and divide it by the second number in the pair and form the table, what will I get?’

‘Try and see.

1 (divided by) 2= 0 R 1
4 (divided by) 3 =1R 1
9 (divided by) 4 =2 R 1
16(divided by) 5=3 R 1 and so on.

Oh what a nice pattern! The square of a number when divided by the consecutive greater number leaves the remainder 1.

‘Can’t you use successor in the place of consecutive greater number?’

‘Oh! Yes. The square of a number when divided by the successor of the number leaves the remainder 1.

‘What about the quotients?’

‘The quotient is the predecessor of the square root, that is, the number taken for squaring.

‘This is marvellous! O.K. Suppose you divide the square of the second number in a pair by the first number in the pair. Will you get a similar pattern?’

‘Let me work out and see:

4	**4**	**5**	**6**	**7**	**8**
1) 4	**2) 9**	**3) 16**	**4) 25**	**5) 36**	**6) 49**
4	**8**	**15**	**24**	**35**	**48**
0	**1**	**1**	**1**	**1**	**1**

‘Except the first pair (1,2), the other pairs show up a similar pattern. The remainder is 1 and the quotient is two more than the first number which is the divisor.’

‘Excellent. Let us meet next weekend.

ROMPING ROUND 2

Triads of Consecutive Natural Numbers

This week my uncle asked me to pick up the Blue Visualisation Chart for my study. This was the Visualisation Chart used by him while talking to members of the Euler Math Club.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**

‘Is it not the same chart used in the math club meeting which I attended with you?’

‘Yes. It is. Can you recall the properties discovered by handling three consecutive numbers?’

‘Oh! Yes. How can I forget them? They are green in my mind. These are the properties:

1. The sum of the first and the last of three consecutive natural numbers is twice the middle number.
2. The product of the end numbers in any triad of consecutive natural numbers is the predecessor of the square of the middle number or the successor of the product of the end numbers in triad of consecutive natural numbers is the square of the middle number.
3. The sum of three consecutive numbers is a multiple of three.
4. The product of three consecutive numbers is a multiple of six.

‘Am I right?’

‘Wonderful. You have given the more beautiful proper- ties first. I congratulate you on your good memory feat. Can you find more properties now?’

‘Suppose I take the product of the first two of the triad of consecutive numbers and divide it by the third number. Will there be a pattern? Let me try.

2 (divided by) 3 = 0 R2
6 (divided by) 4 = 1R2
12(divided by) 5=2R2
20 (divided by) 6=3R2
30 (divided by) 7 = 4 R2
42 (divided by) 8 = 5 R2
56 (divided by) 9 = 6 R2 and so on.

Oh! Yes. I get it. When the product of the first two of a triad of three consecutive numbers is divided by the remaining third number, the reminder is 2.’

‘Fine. Go ahead’

‘Suppose I take the product of the end numbers in a triad of consecutive numbers and divide the product by the middle number. What will happen? Let me see.

3 (divided by) 2 = 1R1
8 (divided by) 3 = 2R2
15(divided by) 4 =4R3
24 (divided by) 5= 4R4
35(divided by) 6 = 5R5
48 (divided by)7 = 6R6   and so on

What a fascinating pattern! The quotient and the remainder are the same when the product of the end numbers in a triad of consecutive numbers is divided by the middle of the triad.'

'How are the quotient and remainder related to the middle number of the triad?'

'Oh! Yes. I was about to examine that. The quotient and the remainder are each predecessor of the middle number'

'Carry on.

'What remains is to examine what happens when the product of the last two numbers in a triad of the product of consecutive numbers is divided by the first number in the triad. Will there be a similar pattern? Let me see

$$\begin{array}{r} 6 \\ 1\overline{)6} \\ 6 \\ \hline 0 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ 2\overline{)12} \\ 12 \\ \hline 0 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ 3\overline{)20} \\ 18 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{r} 7 \\ 4\overline{)30} \\ 28 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{r} 8 \\ 5\overline{)42} \\ 40 \\ \hline 2 \\ \hline \end{array}$$

$$\begin{array}{r} 9 \\ 6\overline{)56} \\ 54 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{r} 10 \\ 7\overline{)72} \\ 70 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{r} 11 \\ 8\overline{)90} \\ 88 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{r} 12 \\ 9\overline{)110} \\ 108 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{r} 13 \\ 10\overline{)132} \\ 130 \\ \hline 2 \\ \hline \end{array}$$

Except for the first two divisions, the others show up a pattern. The remainder is always 2 and so the quotients form the sequence of natural numbers starting from 6. Except for divisions by 1 and 2, in every other division the quotient is 3 more than the division.

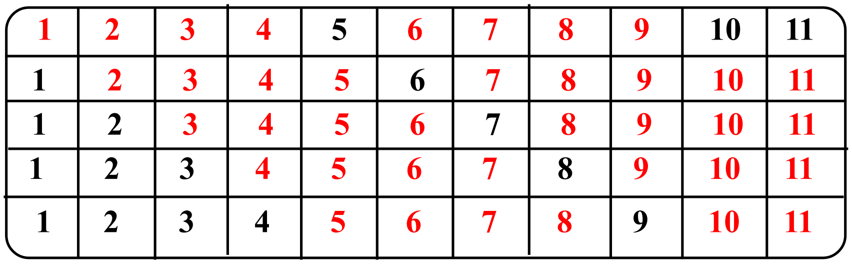
ROMPING ROUND 3

Four Consecutive Natural Numbers

'What do you expect the Visualisation Chart for today to show up?'

'First I had one showing up pairs and then I had another showing up triads. If I were to follow the trend, I should have today the one showing up four consecutive numbers. Right?'

'O.K. Here is the red one:



1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11

Try now.

'I have to try more as numbers are too many in each set of consecutive numbers. Perhaps I may be me lucky here. Suppose I add the two end numbers and then the two middle numbers in each set and compare the two sums. Will there be a pattern? Let me see by setting up a table:

Sum of the end numbers

(1 + 4 =) 5
(2 + 5 =) 7
(3 + 6 =) 9
(4 + 7 =) 11

Sum of the middle numbers

(2 + 3 =) 5
(3 + 4 =) 7
(4 + 5 =) 9
(5 + 6 =) 11 and so on.

The pattern is simple and beautiful. The sum of the end numbers is the same as the sum of the middle numbers in any set of four consecutive natural numbers. Not only that. There is a second pattern. The sums are all odd.

If I do multiplying instead of adding what will I get? Let me form the table and see.

Product of end numbers

(1 x 4 =) 4
(2 x 5 =) 10
(3 x 6 =) 18
(4 x 7 =) 28
(5 x 8 =) 40

Product of middle numbers
(2 x 3 =) 6
(3 x 4 =) 12
(4 x 5 =) 20
(5 x 6 =) 30
(6 x 7 =) 42 and so on.

Yes! there is a pattern. The products are even numbers. The products of the end numbers are each two less than the product of the corresponding middle numbers.

‘Good. Now list the product numbers in their order. Do you see anything arresting?’

‘The product numbers are:

4, 6, 10, 12, 18, 20, 28, 30, 40, 42, etc.

‘Alas! the pattern of this sequence is not easy to find. For the differences rise and fall.

2, 4, 2, 6, 2, 8, 2, 10, 2 and so on.’

‘Is this not fascinating?’

‘Yes. It is.’

‘What do you intend to do next?’

I have not studied the sums got by adding the four numbers in each set.’

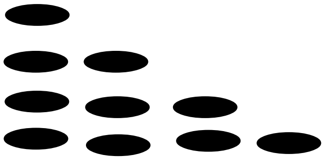
‘O.K. Carry on.’

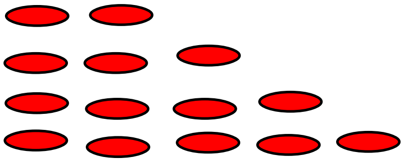
‘Now the sums are
1 + 2 + 3 + 4 = 10
2 + 3 + 4 + 5= 14
3 + 4 + 5 + 6 = 18
4 + 5 + 6 + 7= 22
5 + 6 + 7 + 8 = 26 and so on.

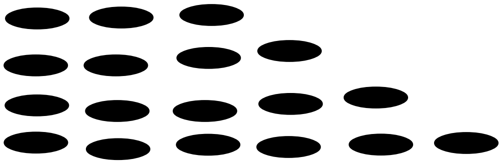
The pattern does not seem to be so interesting. However there is a pattern. The sums form a sequence of numbers starting with 10 and having a common difference of 4.’

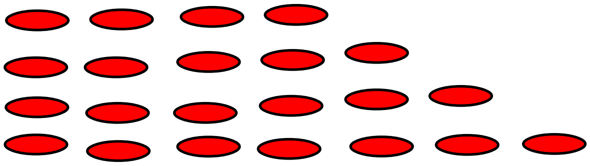
‘Don’t dismiss the sequence so easily. I have something interesting for you here. Set up designs with bottle tops to represent the sum of four consecutive numbers as in the case of triangular numbers.

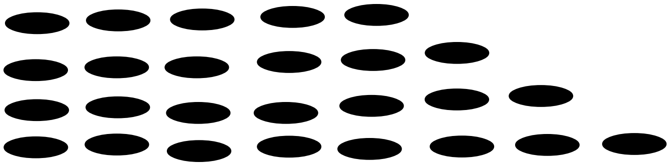
I started doing it.











‘This is what I get.’

‘What shape does each take?’

‘I see. Except the first, the rest take the shape of a trapezium. Right?’

‘Very good. As a matter of fact, these numbers are called trapezoidal numbers.’

‘What about the first?’

‘It is also a trapezoidal number, but a special one, just as square is a special kind of a rectangle?’

‘So trapezoidal numbers are more general than triangular numbers.

‘Am I correct?’

‘You get it. You are fast becoming a Junior Mathematician. Do you think you can find some more patterns in these sets of four consecutive numbers?’

‘Why, I have not studied the products obtained by multiplying the four consecutive numbers, in each set.’

‘I see. Carry on.’

‘The products are:

1 x 2 x 3 x 4 = 24
2 x 3 x 4 x 5 = 120
3 x 4 x 5 x 6 = 360
4 x 5 x 6 x 7 = 840
5 x 6 x 7 x 8 = 1680
6 x 7 x 8 x 9 = 3024 and so on.

‘There seems to be no pattern.’

‘Why not study the predecessor or successor of each product and see if a pattern surfaces?’

‘Oh! I see. I shall try. The predecessors of the products are 23, 119, 359, 839, 1679, 3023 etc. I don’t see any pattern. Let me consider their successors. The successors of the products are 25, 121,361, 841, 1681, 3025, etc. 25 and 121 are square numbers?’

‘Take the Book of Math Tables on the table and refer to the table of squares.

I see. 361, 841, 1681, 3025 are all squares. So the pattern is a deeper one. The product of any four consecutive numbers is one less than a square number.’

‘Very good. You are enjoying yourself. Aren’t you? Well, can you find out how to get the square number in each case? Observe the numbers in each set.’

‘Sure. Now to fix the square numbers, let me try

$25 = 2^2 = (4 + 1)^2$ 1, 4 - 4;
$121 = 11^2 = (10 + 1)^2$ 2, 5 - 10;
$361 = 19^2 = (18 + 1)^2$ 3, 6 - 18 etc.

Oh yes, I got it. The square is obtained by multiplying the successor of the product of the first and the last in each set by itself. Thank you for your question. Now I want to compare the sums got by adding the end numbers and the sums got by adding the middle numbers. Let me make out a table of them.

Sum of the four consecutive numbers	Sum of the two middle numbers	Sum of the two end numbers
10	5	5
14	7	7
18	9	9

and so on.

The pattern is simple. The sum of four consecutive numbers is twice the sum of their middle numbers or twice the sum of their end numbers.’

‘Good. Anything else?’

‘Suppose I add the first and the third and then add the second and the fourth in each set of four consecutive numbers and make out a table of the same. Let me see what emerges when I compare the sums.

Sum of the first and the third number	Sum of the second and the fourth number
4	6
6	8
8	10
10	12

The pattern is simple, but not deep. The sum of the first and the third is two less than the sum of the second and the fourth in any set of four consecutive numbers.

Uncle, I have question. Why should we be having all the time consecutive natural numbers?’

‘Well! Are you getting bored?’

‘Not that. I know now what Visualisation Chart I will get next if you continue the pattern of Visualisation Charts so far presented. There is no surprise. That is what I mean.’

‘Oh! Yes. You will have a surprise when you meet me next weekend.’

ROMPING ROUND 4

Pairs of Consecutive Triangular Numbers

‘Here is the Visualisation Chart for your bout of discoveries today, Did you expect it?’

1	3	6	10	15	21	28	36	45	55	66
1	3	6	10	15	21	28	36	45	55	66
1	3	6	10	15	21	28	36	45	55	66

‘No I didn’t expect it. Thank you. You want me to study pairs of consecutive triangular numbers. Right?’

‘Yes. That is it. Go ahead.’

‘First let me study the sums obtained by adding each pair of consecutive triangular numbers.

The sums are

1 + 4 = 4
3 + 6 = 9
6 + 10 = 16
10 + 15 = 25
15 + 21 = 36
21 + 28 = 49 and so on.

Beautiful! The sum of-two consecutive triangular numbers is a square number

‘Good. Suppose I ask you to give the sum of the tenth pair of triangular numbers without finding them. Can you find it?’

‘Let me try. The sum of the first pair is 2 squared. The sum of the second pair is 3 squared. The sum of the third pair is 4 squared. Now I get it. The sum of the 10th pair is 11 squared or 121. Correct?’

‘Excellent. Go ahead’

‘Let me now study the products obtained by multiplying each pair of consecutive triangular numbers. The products are:

1 x 3 = 3
3 x 6 = 18
6 x 10 = 60
10 x 15 = 150
15 x 21 = 315 and so on.

The pattern is not so inviting. The products are multiples of 3. let me write them as multiples of 3 and see what surfaces.

3 = 1 x 3
18 = 6 x 3
60 = 20 x 3
150 = 50 x 3
315 = 105 x 3 and so on.

Well, I give up. Uncle, why not I consider three consecutive triangular numbers at a time?’

‘Take up the Green Visualisation Chart and proceed.’

1	3	6	10	15	21	28	36	45	55	66
1	3	6	10	15	21	28	36	45	55	66
1	3	6	10	15	21	28	36	45	55	66
1	3	6	10	15	21	28	36	45	55	66

‘Let me start finding the sums by adding the end numbers in each triad of consecutive triangular numbers and then compare the sums with their corresponding middle numbers. This is the table that I get.

Sum of end numbers	Middle numbers
(1 + 6 =) 7	3
(3 + 10 =) 13	6
(6 + 15 =) 21	10
(10 + 21=) 31	15
(15 + 28 =) 43	21
(21 + 36 =) 57	28 and so on.

The pattern is not on the surface. Suppose I divide each sum by the corresponding middle number, what will emerge?

7 (divided by) 3 =2R1
13 (divided by) 6 =2R1
21 (divided by) 10 = 2R1
31 (divided by) 15 = 2R1
43 (divided by) 21 = 2R1
57 (divided by) 28 = 2R1

I get it now. The sum of end numbers in any triad of consecutive triangular numbers is the successor of twice the middle number in the triad’

‘That is superb. Go ahead.

‘Now let me find the products by multiplying the end numbers in each triad of consecutive triangular number and compare the products with their corresponding middle numbers’.

Tabulating them, I get

Product of end numbers	Middle numbers
(1 x 6=) 6	3
(3 x 10=) 30	6
(6 x 15=) 90	10
(10 x 21=) 210	15
(15 x 28=) 420	21 and so on

Yes. I see it. The product of the end numbers in any triad of consecutive triangular numbers is a multiple of 3. Writing the products as multiples of 3, I get

Product	Multiple of 3
6	2 x 3
30	5 x 6
90	9 x 10
210	14 x 15
420	20 x 21 and so on

2, 5, 9, 14, 20 etc have a pattern. The differences found by taking them consecutively two by two are 3, 4, 5, 6 etc, which are natural numbers starting from 3. Also each product has a triangular number as one of its factors.

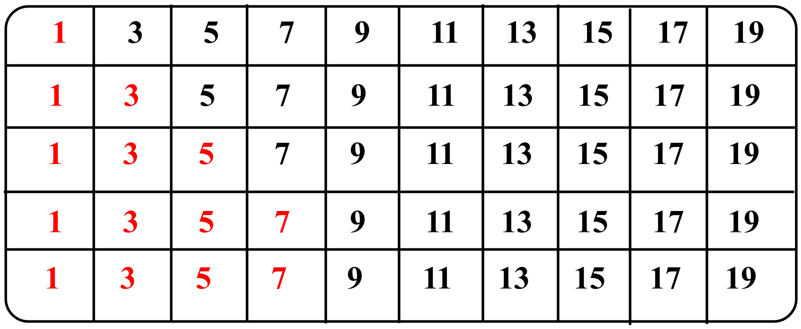
‘Uncle, why should we be considering sets of consecutive numbers all the time?’

‘We need not. You will have another surprise when we meet next.’

ROMPING ROUND 5

Progressive sums of odd numbers

‘Take this Orange Visualisation Chart. What do you notice?’



1	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**
1	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**
1	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**
1	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**
1	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**

‘I see odd numbers in sequence. You want me to consider consecutive odd numbers.’

‘Yes, progressively.’

‘O.K. Let me study their sums first.

$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

and so on.

Another simple and beautiful pattern!

The sum of any number of consecutive odd numbers starting from 1 is a square number.:

‘Good. Now take this Yellow Visualisation Chart. This also deals with odd numbers. See what you are expected to do

1	3	5	7	9	11	13	15	17	19
1	3	5	7	9	11	13	15	17	19
1	3	5	7	9	11	13	15	17	19
1	3	5	7	9	11	13	15	17	19

‘You surprise me, uncle. This is indeed real romping. Well, let me study the sums of odd numbers showing up in each row.

$$3 + 5 = 8$$

$$7 + 9 + 11 = 27$$

$$13 + 15 + 17 + 19 = 64$$

$$21 + 23 + 25 + 27 + 29 = 125$$ and so on.

Wonderful! The sums are all cube numbers.

Oh! I see. 1 is a cube numbers.

That means, the first odd number is a cube number.

he sum of the next two odd numbers is a cube number. The sum of the next three odd numbers is a cube number and so on.’

‘Suppose we call them as a sequence of sets. ‘That is to say, the first set consists of one odd number and it is 1;

the second set consists of the next two odd numbers 3 and 5 adding upto 8;

the third set consists of the next three odd numbers 7, 9 and 11 adding upto 27;

the fourth set consists of the next four odd numbers 13, 15, 17 adding unto 64 etc.

Can you find the first odd number in, say, the tenth set?’

‘You want me to study the odd numbers coming first in these sets. Right?’

‘Yes, you are on the right track. Go ahead.’

‘Let me leave out the first set.

Starting from the second and tabulating, I get

Number of the set	First odd number
2	3
3	7
4	13 and so on.

Dividing the number in the second column by the corresponding number in the first column, I get

This is really wonderful. The remainder is always

$$\begin{array}{r} 1 \\ 2\,\overline{)\,3} \\ 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{r} 2 \\ 3\,\overline{)\,7} \\ 6 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{r} 3 \\ 4\,\overline{)\,13} \\ 12 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{r} 4 \\ 5\,\overline{)\,21} \\ 20 \\ \hline 1 \\ \hline \end{array}$$

1. The quotient is 1 less than the divisor and the divisor is the number of the set. Also the first odd number is the dividend. I get it. Uncle, I can answer your question. The tenth set will start with the odd number 91, because 10 is the divisor, 9 is the quotient and 1 is the remainder.'

'That is a feat indeed. Come on. What will be the sum of odd numbers in the tenth set?'

'I shall give the tenth set itself with its sum. O.K.?'

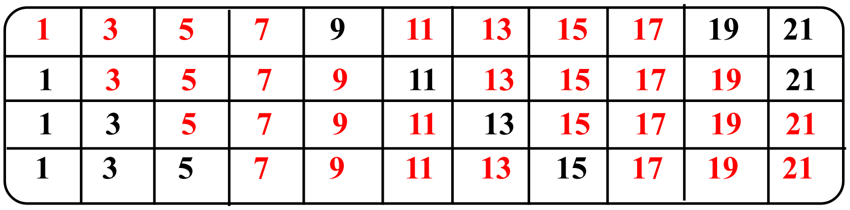
'O.K.?'

91 + 93 + 95 + 97 + 99 + 101 + 103 + 105 + 107 + 109 = 1000

'Splendid'

'So, uncle, I will no more get Visualisation Charts showing equivalent sets of consecutive numbers?'

If you want to take up one of that kind, take this Grey Visualisation Chart.



1	3	5	7	9	11	13	15	17	19	21
1	3	5	7	9	11	13	15	17	19	21
1	3	5	7	9	11	13	15	17	19	21
1	3	5	7	9	11	13	15	17	19	21

'Oh! I see it. You want me to consider sets of four consecutive odd numbers. Let me strike their sums and study them.

1 + 3 + 5 + 7 = 16
3 + 5 + 7 + 9 = 24
5 + 7 + 9 + 11 = 32
7 + 9 + 11 + 13 = 40 and so on

I get it. The sum of four consecutive odd numbers is a multiple of 8. Right?'

'You are seeing the pattern faster now.

'Uncle, do you have visualization charts for square and cube numbers?'

'Well, you can prepare them yourself and consider them the way you fancy'

'Thank you, I shall do so and meet you next week with my discoveries.

ROMPING ROUND 6

Pairs of Consecutive Square Numbers

I took the square numbers first and prepared two Visualisation Charts. The first one is shown below:

1	4	9	16	25	36	49	64	81	100
1	4	9	16	25	36	49	64	81	100
1	4	9	16	25	36	49	64	81	100

This chart is an invitation to study pairs of consecutive square numbers. The sums got by adding pairs of consecutive square numbers do not have an interesting pattern, except that they are all odd. The difference between any two consecutive square is also odd.

As the sums are all different, my curiosity turned to the sequence of numbers obtained as the sums.

5, 13, 25, 41, 61, 85 and so on.

I started writing the differences by taking them two by two. I got

8, 12, 16, 20, 24 and so on.

Striking again their differences by taking them two by two consecutively I got 4, 4, 4,...

This is interesting!

Next I took up the differences obtained from pairs of consecutive square numbers, as they are also different.

3, 5, 7, 9, 11 and so on

Striking the differences of these by taking them two by two consecutively, I got 2, 2, 2,...

I discovered something new. In the case of sum, the differences come to be the same only at the second stage, whereas in the case of differences, the differences come out to be the same in the first stage itself. This was a new experience and more than that, a new opening for my study,

The second Visualisation Chart is shown below:

1	4	9	16	25	36	49	64	81	100
1	4	9	16	25	36	49	64	81	100
1	4	9	16	25	36	49	64	81	100
1	4	9	16	25	36	49	64	81	100

This chart gives the call to study triads of consecutive square numbers.

Adding the end numbers in each triad and comparing the sums obtained with their corresponding middle number, I discovered a deep pattern.

Sum of the end square number in each triad	Middle square under
(1 + 9 =) 10	4
(4 + 16 =) 20	9
(9 + 25 =) 34	16
(16 + 36 =) 52	25 and so on.

Dividing each number in the first column by the corresponding number in the second column, I got

10 (divided by) 4 = 2R2
20 (divided by) 9 = 2R2
34 (divided by) 16 = 2R2
52 (divided by) 25 = 2R2 and so on.

The sum of end square numbers in each triad of consecutive square numbers is two more than twice the middle square number.

My third Visualisation Chart was about cube numbers.

1	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**
1	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**
1	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**
1	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**

This chart directs one’s attention to consecutive cube numbers taken progressively. That is to say, one should consider the first cube, the first two cubes, the first three cubes, the first four cubes and so on.

I worked out their sums and discovered to my pleasant surprise the sums to be squares.

1 + 8 = 9
1 + 8 + 27 = 36
1 + 8 + 27 + 64 = 100 and so on.

The sum of any number of consecutive cube numbers starting from 1 is a square number. Calling them sets, I can say:

the sum of cubes in the set of first two cube numbers is three squared,
the sum of cubes in the set of first three cube numbers is six squared.
the sum of cubes in the set of first four cube numbers is ten squared.

It struck me at once that 3,6,10 etc are triangular numbers. so I found a better way of stating my discovery.

The sum of the first two cube numbers is the square of the second triangular number.

The sum of the first three cube numbers is square of the third triangular number and so on.’

5. TAKING LEAVE

I met my uncle with my findings about squares and cubes. He was immensely pleased with my work. He hugged me. So he and I felt proud. He presented me a book entitled ‘Fun With Numbers’. He said he would be away for six months on a lecture tour abroad and I could resume my weekend visits after his return.

I felt sad that I would be missing him for many weeks.

I asked him why there were no Visualisation Charts dealing with even numbers, prime numbers, multiples of natural numbers, composite numbers etc.

He said, ‘you are now a Junior Mathematician and you have now the knowledge and skill to do romping unaided. So you are free to draw up your own Visualisation Charts dealing with any kind of numbers you like and investigate. I wish you the best of luck.’

I felt myself on top of the world. I asked him to pardon me for not accepting his advice earlier.

‘Don’t worry. It is the case with many,’ he said.

Before taking leave, he asked me to take the Violet Visualisation Chart

$$\begin{gathered} 1 + 2 = 3 \\ 4 + 5 + 6 = 7 + 8 \\ 9 + 10 + 11 + 12 = 13 + 14 + 15 \\ 16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24 \end{gathered}$$

and give the 10th equality without writing equalities one by one. This was different from what I had before and so it was a challenge. I wanted to meet it and enjoy myself.

As before, I started making a methodical study.

Number of the equality	The first number on the L H S
1	1
2	4
3	9
4	16 and so on.

So each equality starts with a square. The first equality starts with the square of one. The second equality starts with the square of two and so on.

So the tenth equality would start with the square of 10, that is, 100.

I have answered his question partially.

Next I should find out the number of numbers that would come on the Left Hand Side (LHS) and Right Hand Side (RHS) of the equality.

Number of the equality	Number of numbers on the LHS	Number of numbers on the RHS
1	2	1
2	3	2
3	4	3
4	5	4 and so on

So the first equality has two numbers on the LHS and one number on the RHS. The second equality has three numbers on the LHS and two numbers on the RHS and so on. So in the tenth equality, the number of numbers on the LHS is 11 and the number of numbers on the RHS is 10.

With these discoveries, I wrote down

100 + 101 + 102 + 103 + 104 + 105 + 106 + 107 + 108 + 109 + 110 =
111 + 112 + 113 + 114 + 115 + 116 + 117 + 118 + 119 + 120

He stood up and shook my hands in appreciation of what I had done before him. He observed that one should be able to raise questions for making discoveries in mathematics. He asked me what other questions one could raise in the pattern shown by the Visualisation Chart.

I said,

(1) What is the first number on the right side of any given equality?
(2) What is the sum of numbers on the LHS of equality?
(3) What is the sum of numbers of the RHS of equality?
(4) What is the last number on the RHS of any equality?
(5) What is the last number of the LHS of any equality?
(6) Can we set up a similar kind of equalities with even numbers, multiples of a number etc?'

'So you have ability and intelligence to raise questions and find answers for them on your own. That is what you· need in doing research. I declare that you are now a research Junior Mathematician.'

'How long will it take to become a Senior Mathematician, uncle?'

'When one develops the art and skill of giving proof or disproof of guesses one makes, one will be on the way to becoming a Senior Mathematician. After all, the properties found so far can only be taken as reasonable or as we say, plausible. Soon after my return, I shall expose you to. proving in mathematics.'

'Well, uncle, I have been seeking pleasures. You have now during these weeks enabled me to seek the joy of life. I cannot thank you fully.' I said with tears in my eyes.

I said 'Good bye' and took leave of him to do romping on my own.

COMPENDIUM OF NUMBER PROPERTIES

1. The sum of any pair of consecutive numbers is an odd number.
2. The difference of two consecutive numbers is one.
3. The product of two consecutive numbers is even.
4. The square of a number when divided by the successor of the number leaves the remainder 1.
5. Except for the pair (1,2), the square of the greater divided by the smaller in any pair of consecutive numbers leaves the remainder 1.
6. The sum of the first and the last of three consecutive natural numbers is twice the middle number.

7. The product of the end numbers in any triad of consecutive natural numbers is the predecessor of the square of the middle number.

8. The sum of three consecutive numbers is a multiple of three.

9. The product of three consecutive numbers is a multiple of six.

10. When the product of the first two of a triad of three consecutive numbers is divided by the remaining third number, the remainder is 2.

11. The quotient and the remainder are the same when the product of the end numbers in a triad of consecutive numbers is divided by the middle number of the triad. The quotient and the remainder are the predecessor of the middle number.

12. Except for division by 1 and 2, in every other division. of the product of the last two numbers by the first number in every triad of consecutive numbers, the quotient is 3 more than the divisor.

13. The sum of end numbers is the same as the sum of middle numbers in any set of four consecutive natural numbers.

14. The product of the end numbers is two less than the product of the middle numbers in any set of four consecutive natural numbers. The sum of consecutive natural numbers starting from 2 or any other natural number, is a trapezoidal number. A triangular number, is trapezoidal number. A triangular number is a trapezoidal number but not vice versa.

16. The product of any four consecutive numbers is one less than the square of the successor of the product of the first and the last of those numbers.

17. The sum of four consecutive number is twice the sum of their middle numbers Or twice the sum of their end numbers.

18. The sum of the first and the third is two less than the sum of the second and the fourth in any set of four consecutive numbers.

19. The sum of two consecutive triangular numbers is a square number.

20. The sum of end numbers in any triad of consecutive triangular numbers is the successor of twice the middle number in the triad.

21. The product of the end numbers in any triad of consecutive triangular numbers is a multiple of 3 and a triangular number is a factor of the product.

22. The sum of n consecutive odd numbers in the nth set starting with the odd number n (n - 1) + 1 is n 3

23. The sum of any number of consecutive odd numbers Starting from 1 is a square number.

24. The quotient is 1 is less than the divisor but the remainder is always 1 when the first number in the nth set of consecutive odd numbers starting with the odd number n (n - 1) + 1 is divided by n.

25. The sum of four consecutive odd numbers is a multiple of 8.

26. The sequence of sums of pairs of consecutive odd numbers leave at the second stage the common difference 4.

27. The sequence of differences of pairs of consecutive odd numbers leave at the first stage the common difference 2.

28. The sum of end square numbers in each triad of consecutive square numbers is two more than twice the middle square number.

29. The sum of the first n cube numbers is the square of the nth triangular number.

30. The sum of n consecutive numbers, starting with (n - 1) 2 is equal to the sum of (n - 1) consecutive numbers following the n consecutive numbers.

Romping in Numberland attempts to show how children can enter the threshold of fascinating world of numbers by using simple skills and imaginative faculties. The system of natural numbers abounds in wonderful patterns. In the book, children discover varied and interesting properties of numbers in a friendly setting.

P K Srinivasan (1924- 2005) is one of the pioneers in popularising mathematics among children. He is author of *Game Way Math, Number Fun with a Calendar, Maths Club Activities, Teaching Aids in Primary School Mathematics* and *How to promote creativity in Learning Mathematics.* He was Curator-Director of Ramanujan Museum & Math Education Centre, Chennai.

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