Static Magnetic Levitation Demonstration

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Abstract

A simple example of a mechanical configuration exhibiting magnetic levitation is presented and analyzed. The configuration uses a pencil which stays horizontally in mid-air under the action of magnetic forces, gravity and contact forces at one tip. The gadget is easy to build from simple material and is an excellent pedagogical tool for illustrating diverse concepts like magnetic dipole interaction, force and torque balance, minima of potential energy, small oscillations and stability etc. The configuration is remarkably stable and static (in contrast to several other examples like levitron) but contains interesting physics. We present a detailed analysis of both its equilibrium configuration as well as stability of small oscillation around the equilibrium. The model makes definite predictions regarding various parameters of the gadget and these are verified directly. This is possibly the simplest configuration exhibiting the static, near neutral equilibrium under the action of gravity, magnetic forces and a contact force at a point.[1]

1 Introduction

Take a look at the photograph in Fig. 1 which shows a pencil suspended in mid air supported essentially by magnetic forces. The configuration consists of a sponge base with four small magnets, each in the shape of a ring. Two more magnets, identical to those in the base are attached to an ordinary pencil. The tip of the pencil touches a light plastic disk cut from a used CD. The configuration is extraordinarily stable; small displacements of the pencil allow for oscillations and in fact the pencil can rotate about its long axis without falling off. The entire configuration is static and can be easily designed with material available around home or school.

[1]A project based on this work won three awards at the Intel International Science and Engineering Fair (Intel ISEF) 2006, including the Third Prize given by the American Association of Physics Teachers and the American Physical Society.

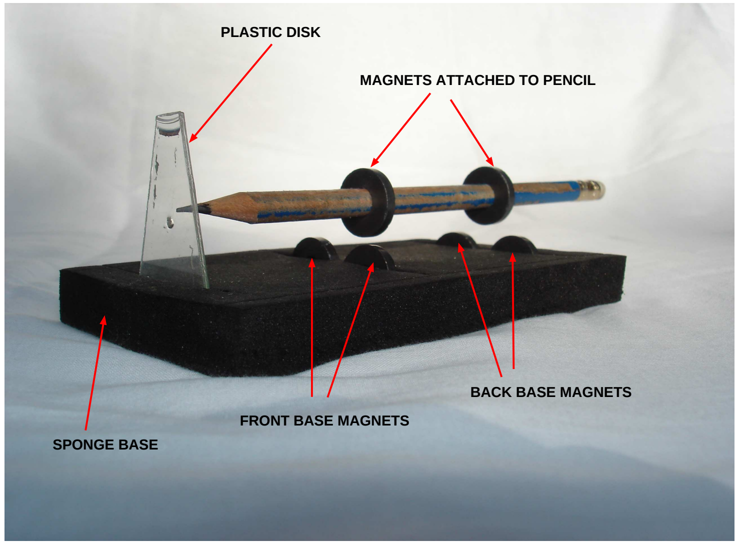

Figure 1: Photo of the actual experimental set-up. There are four ring magnets on the sponge base and two ring magnets on the pencil. The tip of the pencil is in point contact with a vertical plastic disk, cut from a used CD. The pencil levitates in a horizontal position, as shown.

Magnetic levitation, of course, is a well known concept in several practical applications (like levitation based trains) as well as in simple science toys — the most famous of which is the Levitron. But unlike Levitron and many other toys, the configuration in Fig. 1 is *static* and does *not* use diamagnetic properties, which makes it unique. Incidentally, a commercial toy with this design is available with, of course, all the magnets hidden by packaging (see acknowledgement). However, while there are scores of technical papers dealing with Levitron [1], [2] there is virtually no discussion in the literature analyzing the configuration in Fig. 1. As far as the author could find, the only brief reference to this toy is in an article [3] which merely says ".... those with enquiring minds will wish to explore the actual magnetic fields involved.". The purpose of this paper is to do exactly that — provide a full analysis of this configuration. We will see that it uses simple yet ingenious physics to achieve the static and stable configuration. *Further, it can serve as a very useful pedagogical tool in teaching several aspects of physics like magnetic dipoles, equilibrium of forces and torques, stability theory, small oscillations etc.*

The plan of the paper is as follows. The analysis clearly has to address two separate questions. Firstly, how do the force and torque balance in this configuration come about? Secondly, what makes this configuration stable? Section 2 studies the forces and torques involved and derives the the force and torque balance equations. Section 3 discusses the stability of the configuration as also its behaviour at small oscillations around the equilibrium. The last section summarises the conclusions.

2 Force and Torque Balance

To study the forces involved, we will assume that each of the ring magnets is a small dipole. (Comparison of the theory with several models which we made shows that this is an adequate approximation). Given the fact that each of these six magnets used in this configuration can be oriented in two different ways as regards the polarity, the whole configuration can be devised in $2^6 = 64$ ways. We found, by direct experimentation, that out of these 64 configurations, only two independent configurations (not counting two more which are obtained by just reversing all polarities) lead to stable levitation. Both these stable configurations have two crucial features which distinguishes them from the rest : (a) The front pair of base magnets *attract* the front magnet on the pencil and the back pair of base magnets *repel* the back magnet on the pencil. (b) There is a small lateral displacement of the front magnet of the pencil with respect to the front pair of the base magnets. However, the back magnet on the pencil is exactly in the same vertical plane as the back pair of base magnets. At first sight one might have thought that the configuration in Fig. 1 is achieved by both the front and back magnets repelling the magnets on the pencil, thereby balancing it against gravity, with no lateral displacement of the pencil magnet with respect to the base magnet. It is, however, easy to see that such a configuration is totally unstable! We will show in the analysis given below that both the conditions are crucial for stability, thereby also explaining why only two out of 64 configurations work.

To begin with, we need the expression for the force between two magnetic dipoles. Consider two magnetic dipoles of dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$, the displacement vector between their centres being $\mathbf{r}$ in the direction $\mathbf{p}_2$ to $\mathbf{p}_1$. The force $\mathbf{F}$ exerted by $\mathbf{p}_2$ on $\mathbf{p}_1$ is given by

$$\mathbf{F} = \frac{3(\mathbf{p}_1 \cdot \mathbf{r})}{r^5}\mathbf{p}_2 + \frac{3(\mathbf{p}_2 \cdot \mathbf{r})}{r^5}\mathbf{p}_1 + \frac{3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{r^5}\mathbf{r} - \frac{15(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})}{r^7}\mathbf{r} \qquad (1)$$

(For completeness, a derivation of the above expression is provided in Appendix 1.). As to be expected, this expression is symmetric in p_1, p_2; the force exerted by $\mathbf{p}_1$ on $\mathbf{p}_2$ is of course, $-\mathbf{F}$ and is obtained by reversing the direction of $\mathbf{r}$. To fix the notation, let us call the force exerted by the front pair of base magnets on the front magnet on the pencil as $\mathbf{F}_1$ and the force exerted by the back pair of base magnets on the back magnet on the pencil as $\mathbf{F}_2$. (For the magnets which are used, the effect of the front base magnets on the back pencil magnet, etc. are very weak and can be ignored). In addition, there are three more non-magnetic forces we need to consider. First, the weight of the pencil $m\mathbf{g}$ acts at its center of mass in the downward direction. Second, note that the force $\mathbf{F}_1$ has a component in the forward direction along the pencil, as well as a component in the downward direction. (See Fig.2). The small plastic disk, cut from a used CD is required precisely to balance this forward component through an equal and opposite normal reaction force $\mathbf{N}$ which it exerts on the pencil at its tip. Finally, we need to take into account the (possible) frictional force along the plastic disk perpendicular to the above normal force. (We will see that this friction is also necessary for force balance.).

We use a right handed coordinate system, with the $Y-$axis along the pencil, with the positive-Y direction pointing towards its tip; the $Z-$axis points vertically upwards and X-axis in the horizontal plane. Let $\mathbf{r}_1$ be the distance

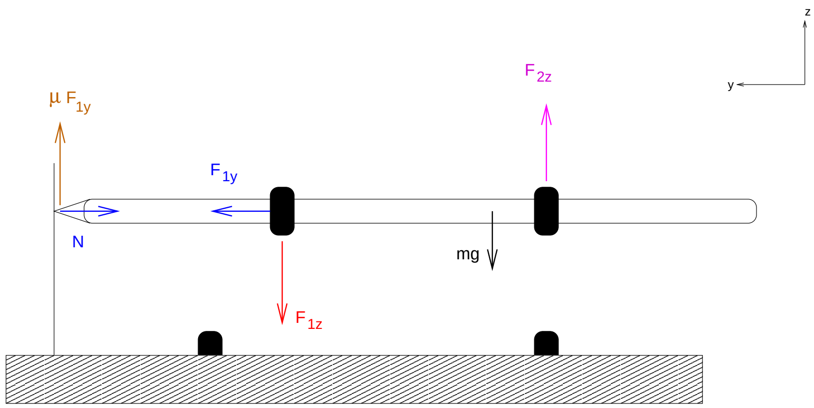

Figure 2: Lateral view of the levitating pencil showing the various forces. Figure is exaggerated for clarity.

between the centres of one of the front base magnets and the front pencil magnet, $\mathbf{r}_2$ the distance between the centres of one of the back base magnets and the back pencil magnet; h the height of the pencil above the base; and $2b$ the distance between the magnets on the base along the $X-$ axis. Finally, let the small lateral displacement of the front magnet on the pencil from the vertical plane of the front magnets be y_0. We can now express all other quantities and the forces in terms of these variables. Since the pencil is equidistant from the two pairs of magnets on the base, the $X-$ component of the displacement vector between any magnet on the base and the corresponding (front or back) magnet on the pencil is b. Hence, $r_1^2 = b^2 + h^2 + y_0^2$ and $r_2^2 = b^2 + h^2$. Since all the magnets are identical, we can take the magnitude of the dipole moment of any one magnet to be p. The dipole moments of all the three back magnets are aligned and we take them to be along the negative $Y-$axis. Since the front base magnets attract the pencil magnet, the dipole moment of the latter should be anti-parallel to that of the front base magnets. We take the direction of the dipole moments of the two front magnets on the base to be the positive $Y-$direction and that of the front magnet on the pencil to be the negative $Y-$direction. Direct experimentation shows that this configuration is stable. (The only other independent configuration that is stable has all the back magnet polarities in the same direction, but the polarities of the front magnets are all reversed. Our analysis applies to both these. Of course, if any configuration works, so will the one obtained by reversing the directions of *all* the dipole moments which is not an independent configuration.)

We will next obtain the components of the forces explicitly. In the front, since $\mathbf{p}_1$ is along the Y-axis, we can obtain the F_{1y} component by taking the dot product of $\mathbf{F}$ in Eq. (1) with $\mathbf{p}_1$ and dividing by p_1. Using $\mathbf{p}_2 = -\mathbf{p}_1 = \mathbf{p}$ and $\mathbf{p}_1 \cdot \mathbf{r} = -y_0 p$ we get:

$$F_{1y} = 2 \times \frac{3p^2 y_0}{r_1^5}\left(3 - \frac{5y_0^2}{r_1^2}\right) = \frac{6p^2 y_0}{r_1^5}\left(3 - \frac{5y_0^2}{r_1^2}\right) \qquad (2)$$

Here, we have multiplied the expression for the force by a factor 2 because we are considering two base magnets. This is a positive quantity showing that the force exerted on the front magnet of the pencil by the two front base magnets has a component in the positive $Y-$direction. Similarly, by taking the dot product of $\mathbf{F}$ with the unit vector along the positive $Z-$ direction, we obtain F_{1z} to be

$$F_{1z} = \frac{6p^2h}{r_1^5}\left(-1 + \frac{5y_0^2}{r_1^2}\right) \tag{3}$$

(Here, too, we have multiplied the expression by a factor 2). This is a negative quantity and hence the force F_1 has a component in the negative $Z-$ direction, i.e., in the downward direction. The $X-$components of the two forces exerted by each of the two front base magnets on the front magnet of the pencil cancel out by symmetry.

The net force along the Y direction will be balanced by the normal reaction at the plastic disk. If the coefficient of friction between the tip of the pencil and the plastic disk is μ, this will generate a frictional force μF_{1y} in the vertical direction.

The situation related to the back magnets is comparatively easy. Since the magnets are in the same vertical plane, by symmetry, the only non-vanishing force component is in the Z-direction. Taking the dot product of $\mathbf{F}$ with $\hat{Z}$ in our general expression, using $\mathbf{p}_2 = \mathbf{p}_1 = \mathbf{p}$, substituting $\mathbf{p}_1 \cdot \mathbf{r} = 0$ and multiplying by 2, we obtain

$$F_{2z} = \mathbf{F}_2 \cdot \hat{\mathbf{z}} = \frac{6p^2h}{r_2^5} \tag{4}$$

This shows that $\mathbf{F}_2$ has a component in the Z direction which is positive so that the force exerted by the back magnets of the base on the back magnet of the pencil is in the upward direction. All the various forces on the pencil are shown in Fig. 2. The force balance in the $Z-$ direction, including the frictional force, requires

$$\frac{6p^2h}{r_1^5}\left(1 - \frac{5y_0^2}{r_1^2}\right) + mg = \frac{6\mu p^2 y_0}{r_1^5}\left(3 - \frac{5y_0^2}{r_1^2}\right) + \frac{6p^2h}{r_2^5} \tag{5}$$

(The direction of the frictional force is chosen with hindsight so that $\mu > 0$; this can be done without loss of generality since the equilibrium conditions will fix the sign, if it is chosen incorrectly.)

For equilibrium, we also need to balance the torques. The requirement of a frictional force is mandatory for torque balance. To see this suppose that there is no friction. Then, we cannot balance the torques taken from the location of the back magnet of the pencil, since both the gravity and the force at the front magnet will be downwards. Hence the frictional force is required for equilibrium. Also note that, if the lateral displacement y_0 vanishes, then the force $\mathbf{F}_1$ would have had no component in the $Y-$direction. Consequently, the normal force exerted by the plastic disk would be zero and there would be no friction making it impossible to balance torques. This provides a proof that the lateral displacement y_0 is needed.

Having realized this, it is, however, more convenient to take moments about the tip of the pencil when the friction does not explicitly contribute to the torque balance equation. If (y_1, y_2, y_3) denote the distances from the plastic disk to the front magnet on the pencil, centre of mass of the pencil and back magnet on the pencil respectively (see Fig 3), the torque balance, about the tip of the pencil

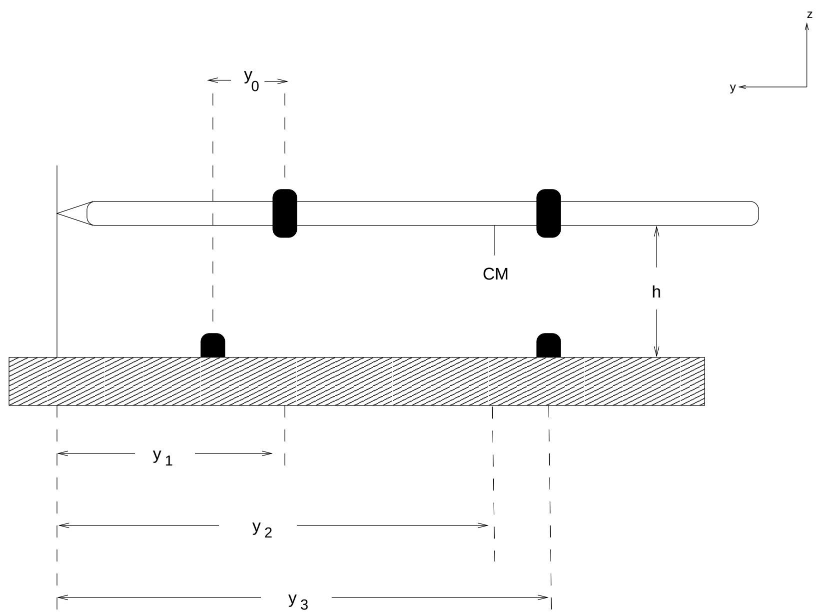

Figure 3: Lateral view of the levitating pencil showing the different lengths used in determining the torque balance.

requires:

$$\frac{6y_1p^2h}{r_1^5}\left(1-\frac{5y_0^2}{r_1^2}\right)+mgy_2=\frac{6y_3p^2h}{r_2^5} \tag{6}$$

The rest of the analysis involves using these equations to obtain the stable configurations. It is convenient to manipulate the Eq. (5) and Eq. (6) and obtain expressions for mg/p^2 and μ. We get:

$$\frac{mg}{p^2}=\frac{y_3}{y_2}\frac{6h}{r_2^5}-\frac{y_1}{y_2}\frac{6h}{r_1^5}\left(1-\frac{5y_0^2}{r_1^2}\right) \tag{7}$$

$$\mu=\frac{hr_1^7}{y_0(3r_1^2-5y_0^2)}\left[\frac{(y_2-y_1)(r_1^2-5y_0^2)}{y_2r_1^7}+\frac{(y_3-y_2)}{y_2r_2^5}\right] \tag{8}$$

By inspection, we find that only the ratios, y_1/y_3, y_2/y_3, and mg/p^2 are relevant for the equilibrium condition. Hence, it is possible to vary the actual distances y_1, y_2, and y_3 as also m and p separately keeping the ratios constant and obtain several different stable configurations.

The typical coefficient of friction for the pencil tip on the plastic disk is about 0.6. Also, mg/p^2 for this system we experimented with is about 0.15 cm^{-4} in C.G.S. units. For the set up shown in Fig. 1 the observed values of y_1, y_2, y_3, b, h and y_0 are 5cm, 9 cm, 10 cm, 1.4 cm, 1.7 cm, and 0.5 cm respectively. Substituting these values into the right hand sides of Eqs.(7), (8) we get $mg/p^2 = 0.15$ and $\mu = 0.57$ showing that the the equations hold.

Before we proceed further, we point out an important fact. Although the value of mg/p^2 is a constant for any given system, the value of μ is not constant and depends on the value of the other forces (like F_{1y}, F_{1z}, etc.). As these forces change, (by varying distances like b, h, etc.) the value of μ will correspondingly change to maintain equilibrium. However, μ cannot assume a value beyond μ_s, the maximum coefficient of static friction for the two surfaces (the tip of the pencil and the plastic disc). The situation is analogous to the case of a block at rest on an inclined plane due to the balance of the normal reaction force exerted by the plane, the weight of the block, and the static frictional force on the block. The block is at rest for a range of angles of the plane (see, for example, [5], pages 132-135) which implies that the coefficient of static friction between the block and the plane can assume a range of values.

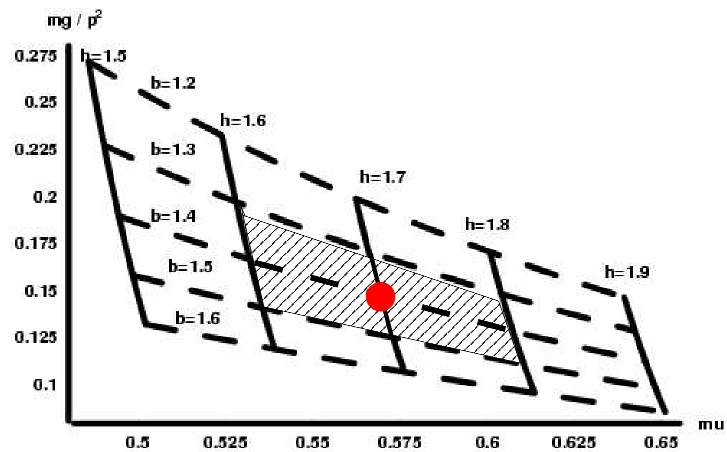

Figure 4: mg/p^2 is plotted against μ for different values of h and b.

To study these aspects, we have plotted mg/p^2 against μ for different values of h and b in Fig 4. The dashed curves are for b = constant with $b = 1.2$ cm, 1.3 cm, 1.4 cm, 1.5 cm, and 1.6 cm with the value of b increasing from top to bottom. The unbroken curves are for h = constant with $h = 1.5$ cm, 1.6 cm, 1.7 cm, 1.8 cm, and 1.9 cm with the value of h increasing from left to right. The intersection of curves pick out the values for mg/p^2 and μ such that Eq.s. (5), (6) are satisfied. We see that for a given value of mg/p^2 (i.e. for a given system) and for sensible values of h and b (possible experimental parameters) the value of μ ranges from 0.5 to about 0.65.

3 Stability Analysis

The most amazing and striking aspect of this configuration is its stability. The pencil can rotate about its long axis without falling off and also perform small oscillations about its equilibrium position. What is more, one notices that it is not at all difficult to place the pencil in its equilibrium position of height (This is in sharp contrast to the Levitron in which making the top spin properly is a work of art!). This implies that the pencil is able to find its own equilibrium position and that the equilibrium should be fairly neutral. We now analyze the stability of the configuration.

To do this, we need introduce some suitable potential energy function the minimum of which gives the equilibrium force balance. Then the behaviour of the potential near the minimum will allow us to study the stability of the configuration. The potential energy of interaction due to magnetic dipoles and the gravitational potential energy are easy to obtain; but for consistency, we also need to introduce a pseudo-potential such that its gradient in the vertical direction gives the frictional force. We shall now work out the details for each of the three directions separately.

3.1 Stability in the Z-direction

Let us first consider the stability along the vertical direction. The force balance along the vertical direction is given by Eq.(5) and we need to find a potential energy function U_{tot} such that $\partial U_{tot}/\partial h = 0$ leads to this equation. This potential can be written as $U_{tot} = mgh + U_{mag} + U_{fric}$ where the first one is gravitational potential energy, the second one is due to magnetic interaction, and the third one is chosen so that its gradient gives the frictional force term in Eq.(5). Of course, friction is not a conservative force but as far as the force balance equation goes, it is convenient to introduce this pseudo potential U_{fric} just for convenience in the analysis.

It is possible to derive the potential energy of interaction between two dipoles of dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ with their centres separated by $\mathbf{r}$. (See [6], pg. 101.) The result is

$$U = \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{r^3} - \frac{3(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})}{r^5} \tag{9}$$

On substituting the relevant values for $\mathbf{p}_1, \mathbf{p}_2, r$ and $\mathbf{p}_1 \cdot \mathbf{r}$, we obtain the expression for the magnetic potential energy our system:

$$\frac{U_{\text{mag}}}{p^2} = -2\left[\frac{1}{r_1^3} - \frac{3y_0^2}{r_1^5}\right] + \frac{2}{r_2^3} \tag{10}$$

The first term is for magnets in the front and the second term is for the back magnets; the factor 2 is again for taking care of two pairs of interaction.

The negative gradient of U_{mag} will give the magnetic force in Eq.(5) while the negative gradient of mgh will give the gravitational force. For consistency, we need to add a pseudo potential energy U_{fric} so that $-\partial U_{fric}/\partial h$ gives the frictional force in Eq,(5). This is determined by the integral:

$$\begin{aligned} \frac{U_{\text{fric}}}{p^2} &= -\int \mu F_{1y} dh \\ &= \frac{-6\mu y_0 h}{3q^6 r_1^5}\left[3q^4(3q^2 - 5y_0^2) + 2h^4(3q^2 - 4y_0^2) + 5h^2(3q^4 - 4q^2 y_0^2)\right] \end{aligned} \tag{11}$$

where $q^2 = b^2 + h^2$. (The second line is obtained by explicit integration of μF_{1y}, using the expression in Eq.(2). The integral is elementary but a bit tedious.) Strictly speaking, the μ in the above equation is not a constant, but for the purpose of illustration, we have treated it (approximately) as a constant at some effective value; observations show that it is approximately 0.57. Even though μ varies with h [as shown in Fig. 4], its range of variation is limited for the range of values of h that we are principally interested in. Taking into account the gravitational potential energy as well, we obtain for the total potential energy of the system U_{tot}

$$\frac{U_{\rm tot}}{p^2} = -2\left[\frac{1}{r_1^3} - \frac{3y_0^2}{r_1^5}\right] + \frac{2}{r_2^3} + \frac{mgh}{p^2} + \frac{U_{\rm fric}}{p^2} \tag{12}$$

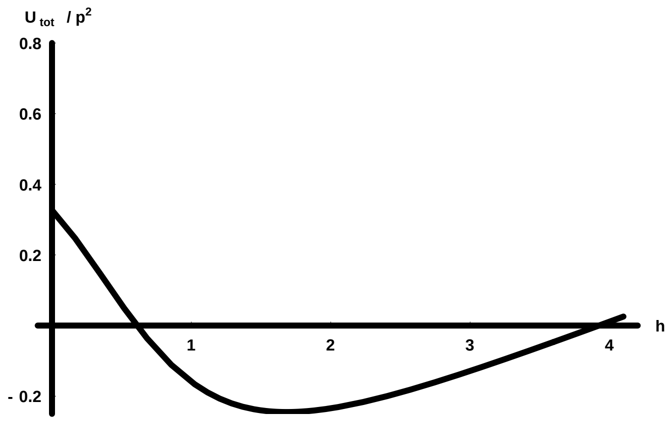

Figure 5: The variation of the potential energy with the height of the pencil above the base when all other parameters are held fixed. The minimum corresponds to the stable equilibrium configuration.

Figure 5 shows the variation of potential energy $U_{\rm tot}$ with respect to h. It can be seen that around the minimum value of the potential energy the curve is reasonably flat (with the minimum value changing by about 10 per cent when h varies by 20 per cent), indicating an almost neutral minimum. This explains the fact that in the model the pencil is able to move slightly up and down and still maintain equilibrium.

3.2 (In)Stability in the y-direction

Let us next consider what happens to the system if the pencil is displaced along the Y-direction, in the absence of the plastic disk. (Of course, the plastic disk is sufficiently rigid that the displacement along Y-axis is quite irrelevant in its presence.) To do this we need to consider the magnetic potential energy U_{mag} of the configuration with the pencil displaced by a distance y in the $Y-$direction, i.e., along the length of the pencil. (The mgh does not change for displacements along Y-axis and there is no friction in the absence of the plastic disk etc.). The expression, which is easily obtained from the original one in Eq.(10) is:

$$\begin{aligned}\frac{U_{\rm mag}(y)}{p^2} &= -\frac{2}{(b^2+h^2+(y+y_0)^2)^{3/2}} + \frac{6(y+y_0)^2}{(b^2+h^2+(y+y_0)^2)^{5/2}} \\ &+ \frac{2}{(b^2+h^2+y^2)^{3/2}}\end{aligned} \tag{13}$$

For the equilibrium value of h, the variation of the potential energy with respect to y is shown in left panel of Fig. 6. Obviously, there is no turning point at

$y = 0$ showing that there is a magnetic force along the pencil. We already know this and the plastic disk provides the normal reaction force to balance this. The Taylor series expansion for the potential energy as a function of y around $y = 0$ is $U_{mag}/p^2 = 0.29 + 0.14y - 0.09y^2 + ...$ in C.G.S. units. From the linear term, we find that the magnitude of the force exerted in the $Y-$direction is $0.14p^2$ (in C.G.S. units). This, of course, should agree with the value of F_{1y}, which it does. (The left panel of Fig. 6 shows that the U_{mag} does have a minimum at $y = -1.06$ cm; however, for this value of y, there is no minimum for the potential energy with respect to h and hence it is not of physical relevance. The right panel of Fig. 6 shows the variation of U_{mag} with respect to h for $y = -1.06$ cm by a dashed curve which has no minimum. For comparison, we have also re-plotted the curve for $y = 0$ shown earlier in Fig. 5 which has a stable minimum.)

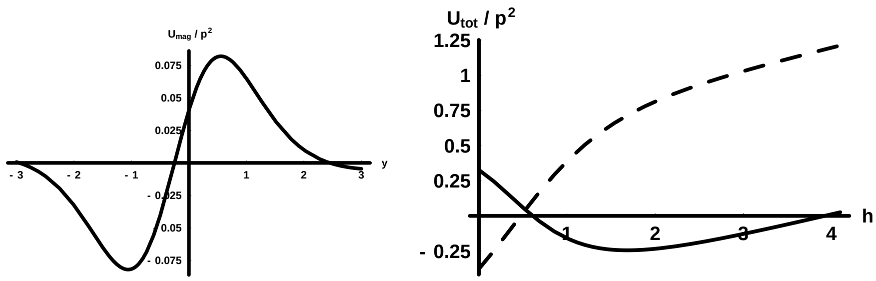

Figure 6: The left panel shows the variation of the magnetic potential energy of interaction with displacements along the $Y-$ direction. Clearly, there is no minimum at $y = 0$ showing that there is a forward component to the magnetic force. The minimum at $y = -1.06$ cm is of no relevance since there is no minimum for h for this value of y. This is indicated in the right panel by plotting the variation of the potential along the vertical direction. The dashed curve, which has no minimum is for $y = -1.06$ cm (in contrast to the unbroken curve which is for $y = 0$.)

3.3 Stability in the x-direction

The pencil can perform small oscillations in the $X-$direction, though it becomes unstable if the displacement is too large. To analyze these oscillations in the $X-$direction, we need to obtain expressions for the potential energy of the system when the back end of the pencil is displaced by a small distance, keeping the tip fixed. (Once again, we only need to consider U_{mag} since friction and gravity are irrelevant for oscillation in the X-direction.) Let the resulting displacement of the back magnet be x under this displacement. Hence the displacement of the front magnets will be $(y_1/y_3)x$. The expression for the potential under this condition is

$$\begin{aligned} \frac{U_{\rm mag}(x)}{p^2} &= \frac{1}{((b+x)^2+h^2)^{3/2}} + \frac{1}{((b-x)^2+h^2)^{3/2}} \qquad (14) \\ &- \frac{1}{((b+(y_1/y_3)x)^2+h^2+y_0^2)^{3/2}} - \frac{1}{((b-(y_1/y_3)x)^2+h^2+y_0^2)^{3/2}} \end{aligned}$$

$$+ \quad \frac{3y_0^2}{((b+(y_1/y_3)x)^2+h^2+y_0^2)^{5/2}} + \frac{3y_0^2}{((b-(y_1/y_3)x)^2+h^2+y_0^2)^{5/2}}$$

(Note that the relative distance along the X axis changes to $(b+x), (b-x)$ at the back end etc. under this displacement. Hence each of the four pairs of interaction has to taken into account separately in this case.) Fig. 7 shows the variation of the potential energy with respect to x. We find a clear minima at $x=0$, as to be expected and an instability occurring for large displacements of around the length scale b.

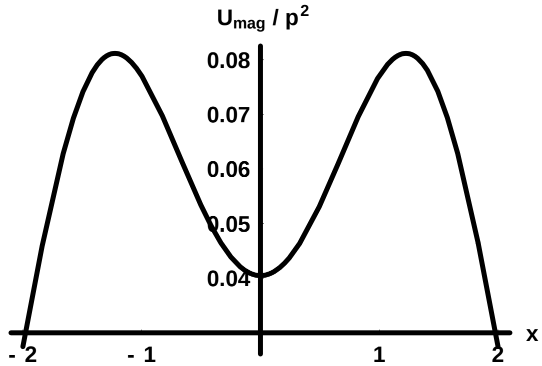

Figure 7: The variation of magnetic potential energy when the back end of the pencil is displaced laterally, along the X- axis, keeping the position of the tip fixed. The minimum at $x=0$ shows that the pencil can perform small oscillations as a rigid body with a frequency, calculated in the text. It is also clear that an instability will occur for large displacements.

As a simple check on the model, one can compute the frequency of oscillation of the pencil about its equilibrium position in the $x-$direction under this condition. The Taylor series expansion of U_x as a function of x about $x=0$ is

$$\frac{U_{\rm mag}(x)}{p^2} \approx U_0 + \frac{k}{2p^2}x^2 \approx 0.30 + 0.06x^2 + ... \tag{15}$$

The term quadratic in x in Eq. (15) is $0.06x^2$. If we treat the pencil as a rigid body performing small angular oscillations about its tip, the equation of motion of the pencil is given by

$$I\frac{d^2\theta}{dt^2} = -y_3(kx) = -\frac{1}{2}y_3^2k\theta \tag{16}$$

where $y_3(kx)$ is the torque due to the force kx, the $I=(1/3)ml^2$ is the moment of inertia of the system about the tip of the pencil and $\theta=(2x/y_3)$. Hence,

$$\omega^2 = \frac{3}{2}\frac{ky_3^2}{ml^2} = 3g\frac{k}{2p^2}\frac{p^2}{mg}\left(\frac{y_3}{l}\right)^2 \tag{17}$$

We known from the series expansion in Eq.(15) that $k/2p^2 = 0.06$ while from the equilibrium condition for our system $mg/p^2 = 0.15$ with both in CGS units.

Substituting the numerical values, we obtain $\omega = 18.54\ \mathrm{s}^{-1}$. The corresponding time period of oscillation is $T = (2\pi/\omega) = 0.34$ seconds. This time period roughly agrees with observations.

Similarly, it is possible to compute the frequency of small oscillations of the pencil in the vertical i.e $z-$ direction. We expand U_{tot} in a Taylor series about the equilibrium value of h. This yields

$$\frac{U_{\mathrm{tot}}(x)}{p^2} \approx U_0 + \frac{k}{2p^2}x^2 \approx -0.24 + 0.126(h-1.68)^2 + ... \tag{18}$$

The term quadratic in $(h-1.68)$ is $0.126(h-1.68)^2$. Hence, approximating the small oscillations of the pencil as simple harmonic motion, we obtain $k/2p^2 = 0.126$. Again, from the equilibrium condition of the system, $mg/p^2 = 0.15$. Substituting these numerical values in the expression $\omega^2 = k/m$, we obtain $\omega = 40.6\ \mathrm{s}^{-1}$. The corresponding time period of oscillation is $T = (2\pi/\omega) = 0.15$ seconds. This time period also agrees with the experimental observations.

4 Conclusions

In magnetostatics, just as in electrostatics, one can introduce a scalar potential Φ_M such that $B = -\nabla\Phi_M$ and $\nabla^2\Phi_M = 0$ in empty space. It follows that one cannot have stable equilibrium [4] for points in empty space in a purely magnetostatic configuration. All examples of magnetic levitation circumvent this difficulty by some extra feature. The usual procedure is to have non-static configurations or moving parts (like in the Levitron in which angular momentum of the top provides the stability). One can also circumvent the difficulty by having rigid bodies (which are no longer confined to a point) and introducing non magnetic forces. Of course, the elegance and utility of such configurations depends on minimum use of non-magnetic forces.

In that respect, the example studied here is truly elegant in its simplicity. The rigid body that is levitated is as simple as one can imagine — namely a linear one dimensional pencil, second in simplicity only to a point object. It is also clear that to balance the gravity stably one requires at least four magnets on the base and two on the pencil. Our analysis shows that simplest of such configurations one might have thought of building — with all base magnets repelling the pencil magnets, thereby balancing the weight — is unstable. *The first innovation in the design of this gadget lies in displacing one of the magnets on the pencil slightly with respect to the base and also making the front base magnets attract the magnet on the pencil.* Once this is done, we require extra forces to compensate the force along the pencil and to provide proper torque balance. *The second innovation lies in realising that both these can be supplied by just a contact force at the tip of the pencil by resting it on a rigid, vertical, plastic sheet.* Under such conditions the configuration almost works itself out uniquely.

Our analysis shows exactly how the force and torque balances come about in such a configuration. What is more, the equilibrium is fairly neutral. one does not have to struggle to set it up (unlike e.g., in the case of the Levitron). As a consequence, the equilibrium is remarkably stable to perturbations. We showed explicitly that the potential energy has a broad minima as far as variation in the vertical direction is concerned. Since levitation is essentially about motion in vertical direction this takes care of one's major concern. Further, the potential has a simple minima for horizontal rigid body oscillations of the pencil with one end fixed. Here, large displacements do cause instability but for small oscillations the force is harmonic. The model allows us to determine the period of small oscillations about the minimum which agrees reasonably with the observations.

There are several further directions in which this work can be extended.

(a) To begin with, it is rather surprising that the dipole approximation works reasonably well in modeling the interaction between the magnets. The separation between the magnets are not too large compared to their size and the magnets themselves are of ring shape so that one would have expected some extra corrections. It is in principle possible to determine the magnetic field of the ring shaped configuration but the result, unfortunately is not expressible in terms of elementary functions (and requires elliptic functions). It will be worth investigating why the corrections to dipole are small.

(b) Also, it is quite possible that one could construct similar configurations possibly even without the contact forces e.g. by using three pencils at 120 degree separation touching at one point. These issues are under investigation.

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(b) American Association of Physics Teachers and the American Physical Society Prize for projects in Physics, Third Prize ($300).

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Appendix 1

This Appendix 1 derives the expression for force for dipole interaction. For ease of comparison with text books, we will derive it for electric dipole; since magnetostatics has the same structure as electrostatics, the same result holds for magnetic dipoles.

The potential energy of interaction between two dipoles of dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ is given by

$$U = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{x}|^3} - \frac{3(\mathbf{p}_1 \cdot \mathbf{x})(\mathbf{p}_2 \cdot \mathbf{x})}{|\mathbf{x}|^5} \tag{19}$$

where $\mathbf{p}_1$ is located at $\mathbf{x}$, $\mathbf{p}_2$ at the origin and the vector $\mathbf{x}$ points from $\mathbf{p}_2$ to $\mathbf{p}_1$. (See [6], pg. 101.) This expression may be written as

$$U = p_1^i p_2^j \partial_i (x^j/|\mathbf{x}|^3) = p_1^i p_2^j \left[\frac{\delta_{ij}}{|\mathbf{x}|^3} - \frac{3x^i x^j}{|\mathbf{x}|^5} \right] \tag{20}$$

where the index $i, j, ... = 1, 2$,etc. denote the components of a vector. We are also using the convention that any index that is repeated in an expression is summed over; and the identities:

$$\partial_a x^b = \delta_a^b; \quad \partial_a |\mathbf{x}| = (x^a/|\mathbf{x}|) \tag{21}$$

We differentiate the Eq. (20) with respect to a third index k to obtain the (k-th component of) the force exerted by $\mathbf{p}_2$ on $\mathbf{p}_1$:

$$\begin{aligned} F_k &= -\partial_k U = -p_1^i p_2^j \partial_k \left[\frac{\delta_{ij}}{|\mathbf{x}|^3} - \frac{3x^i x^j}{|\mathbf{x}|^5} \right] \\ &= -p_1^i p_2^j \left(-\frac{3}{|\mathbf{x}|^4} \frac{x_k}{|\mathbf{x}|} \delta_{ij} - \frac{3}{|\mathbf{x}|^5} (\delta_{ik} x_j + \delta_{jk} x_i) + \frac{15}{|\mathbf{x}|^6} \frac{x_k}{|\mathbf{x}|} x_i x_j \right) \end{aligned} \tag{22}$$

In standard vector notation this becomes:

$$\mathbf{F} = \frac{3\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{x}|^5} \mathbf{x} + \frac{3\mathbf{p}_2 \cdot \mathbf{x}}{|\mathbf{x}|^5} \mathbf{p}_1 + \frac{3\mathbf{p}_1 \cdot \mathbf{x}}{|\mathbf{x}|^5} \mathbf{p}_2 - \frac{15(\mathbf{p}_1 \cdot \mathbf{x})(\mathbf{p}_2 \cdot \mathbf{x})}{|\mathbf{x}|^7} \mathbf{x} \tag{23}$$

This is the expression used in the text.

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